Abstract of the doctoral thesis

titled: **„The ideals of nowhere dense sets in topologies on the set of natural numbers"**

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This Ph.D. thesis is devoted to the study of the ideals of nowhere dense sets in topologies on the set of natural numbers. The problems discussed in the dissertation, lying at the intersection of set theory, combinatorics, number theory and topology, provide an excellent opportunity to observe the interpenetration of seemingly separate areas of mathematics.

One of the explored issues is the concept of Marczewski-Burstin representability, which has been studied by, among others, M. Balcerzak, A. Bartoszewicz, K. Ciesielski and P. Koszmider. In 2001, M. Balcerzak, A. Bartoszewicz, J. Rzepecka and S. Wroński observed that the scheme defining the family of nowhere dense sets in a given topology turns out to be interesting also if the family of nonempty open sets is substituted with any family $\mathcal{F}$ of nonempty subsets of an arbitrary set X. The family:

$$S^0(\mathcal{F}) = \{A \subseteq X :\ \forall_{U\in\mathcal{F}}\ \exists_{V\in\mathcal{F}}\ V \subseteq U \setminus A\}$$

is therefore a generalization of not only the notion of nowhere dense sets, but also the concept of classical Marczewski null sets (s^0), dating from 1935. The ideals that can be written as $S^0(\mathcal{F})$ for a certain family $\mathcal{F}$ are called MB-representable. In the thesis, for ideals on the set of natural numbers, we introduce a more restrictive variant of the notion of MB-representability, additionally requiring the family of subsets of $\mathbb{N}$ representing a given ideal to be countable – we say that such ideals are MB-countably-representable (in short: $\mathcal{MBC}$). We examine their properties and show the relationship with the ideals having a topological representation, introduced by M. Sabok and J. Zapletal. We also consider the concept of extendability to $\mathcal{MBC}$ ideals.

The main purpose of the dissertation, however, is to examine the ideals of nowhere dense sets in topologies on $\mathbb{N}$ with the bases consisting of infinite arithmetic progressions satisfying certain conditions. Several examples of such topologies can be found in the literature. In 1955, H. Furstenberg introduced a topology with the use of which he presented an elegant topological proof of the existence of infinitely many prime numbers. In 1959, S. Golomb demonstrated a similar proof, using the topology defined in 1953 by M. Brown. In 1969, A. M. Kirch modified the definition of this topology in order to obtain some additional properties of the set $\mathbb{N}$ equipped with it. We will also consider the division topology defined in 1993 by G. B. Rizza and the common division topology introduced by P. Szyszkowska (Szczuka) in 2013.

All the above-mentioned topologies have recently been extensively studied by P. Szyszkowska (Szczuka), and some of them also by, among others, T. Banakh, J. Mioduszewski, S. Turek, as well as P. L. Clark, N. Lebowitz-Lockard, F. Marko, Š. Porubský and many other mathematicians from around the world. It is worth noticing that a few of these topologies – constituting nontrivial but elementary examples and counterexamples for many topological properties – are also discussed in the classical book "Counterexamples in Topology" by L. A. Steen and J. A. Seebach.

In the thesis, we define the ideals of Furstenberg, Golomb, Kirch, Rizza and Szyszkowska as the ideals of nowhere dense sets in the mentioned topologies. First, we discuss their basic properties important from the point of view of the theory of ideals on countable sets. We also deal with, e.g., the property FinBW introduced in 2007 by R. Filipów, N. Mrożek, I. Recław and P. Szuca for ideals on $\mathbb{N}$ – we present a new characterization of ideals with the property FinBW in order to use it to obtain some results for the ideals of nowhere dense sets in the studied topologies. This property may also be connected to the extendability to summable ideals and the Riemann property (related to the classical theorem on conditionally convergent series), which have been researched, e.g., by R. Filipów, P. Klinga, A. Nowik, P. Szuca and W. Wilczyński. We also give a negative answer to a question recently posed by C. Uzcátegui in a survey article on current topics concerning ideals on countable sets, showing that the set of natural numbers equipped with Rizza's topology is a countable topological space without isolated points, in which the ideal of nowhere dense sets is $\mathcal{MBC}$, but is not isomorphic to $\mathrm{NWD}(\mathbb{Q})$.

The major part of the thesis, however, is an exploration of the relations between the studied ideals. We prove the inclusions occurring between them or we justify their lack by constructing examples of sets distinguishing particular ideals. We also answer some questions posed by A. Nowik and P. Szyszkowska.

One of the chapters of the dissertation is devoted to the consideration regarding the topologies $\mathcal{T}_m$, for $m \in \mathbb{N}$, recently introduced by P. Szyszkowska – it is a generalization of the definition of the Kirch's topology, which, according to this convention, constitutes the family $\mathcal{T}_1$. We propose a further generalization of the above definition, introducing topologies $\mathcal{T}_{(\alpha_n)}$ for sequences $(\alpha_n)_{n\in\mathbb{N}}$ of nonnegative integers. We investigate the possible inclusions between the ideals of nowhere dense sets $\mathcal{I}_{(\alpha_n)}$ in these topologies – e.g., we give a characterization when $\mathcal{I}_{(\alpha_n)} \subseteq \mathcal{I}_{(\beta_n)}$. The obtained results allow us to draw conclusions for the ideals of nowhere dense sets in topologies $\mathcal{T}_m$ – we get, in particular, a countable chain of topological ideals, where $\mathcal{I}_m \subsetneq \mathcal{I}_{m+1}$ for each $m \in \mathbb{N}$. We also show that among the ideals $\mathcal{I}_{(\alpha_n)}$ one can find an $\subseteq$-antichain of size $\mathfrak{c}$.

In the further part of the thesis, we discuss the relationships between selected well-known ideals playing an important role in number theory and combinatorics, such as the ideal of sets of asymptotic density zero $\mathcal{I}_d$, the summable ideal $\mathcal{I}_{\frac{1}{n}}$, the van der Waerden ideal $\mathcal{W}$ and the Hindman ideal $\mathcal{H}$, as well as the topological ideals thoroughly researched in the dissertation: the ideals of Furstenberg, Golomb, Kirch, Rizza and Szyszkowska. In this way, we summarize how the considerations undertaken in the thesis fit into the classical results that have been known for years.