SUMMARY OF PHD DISSERTATION IN ENGLISH

A. Motivation and main goals of PhD thesis

Quantum mechanics enables an experimentally provable description of atomic and solid state physics, optics, open systems and elementary particles. Its formalism is based on postulates leading to one- and many-body correlations that lack characterization in terms of classical probability theory. In this sense classical probability theory is restricted to describing macroscopic systems. Due to the interaction with an environment, the quantum theory of evolution of systems and their probability distributions are reduced to their classical analogues in a process called decoherence. In the context of the numerous applications of quantum correlations, e.g. in numerical algorithms showing exponential speed-ups with respect to their classical counterparts, in cryptography or in the simulation of quantum systems, the idea of protecting quantum correlations from destruction in the presence of external noise gains, apart from a pure cognitive motivation, a strong practical justification. On the other hand, the commonness of decoherence in nature prompts a search for physical systems that exploit the interaction with an environment as a catalyst of microscopic processes. This dichotomy is reflected in the structure of the thesis. We present proofs and procedures of quantum communication that can be implemented in quantum cryptography and computing. They are based on quantum error correction schemes aimed at minimizing the effect of decoherence. On the other hand, inspired by processes benefiting from the quantum-to-classical transition occurring in nature, we analyze the possibility of employing quantum dot systems that can be realized experimentally in magnetometry.

Establishing quantum communication between spatially separated subsystems in the presence of noise is the goal of quantum repeaters idea [1]. It is based on generating entanglement (a type of quantum correlations, [2]) between separated nodes of a one-dimensional network, which is obtained by exploiting local quantum correlations shared by subsystems placed on neighboring nodes. Long-distance quantum communication in one dimension can be established by applying quantum entanglement purification and swapping protocols, with the cost of the number of systems at specific nodes increasing logarithmically with the size of a network. The requirement for quantum memories able to store the increasing number of systems per node follows. In order to eliminate this inconvenience, one can use a three-dimensional quantum network taking advantage of global quantum correlations and quantum error correction protocols [3]. The aim of work [A] is to deliver a lacking proof of quantum communication in two-dimensional quantum networks with demand that procedures do not rely on long-term quantum memories. The reasoning there is based on considering the fidelity of quantum state encoding into a one-dimensional quantum error correcting code. Furthermore, in the manuscript we present a simple communication scheme in three dimensions, based on an encoding procedure of an unknown quantum state into a Kitaev planar code [4]. In manuscript [B] the encoding procedure was generalized to Calderbank-Stean-Shor (CSS) [5–7] topological quantum error correcting codes. It can be applied in the realization of quantum memories and universal quantum computing based on magical state distillation [8].

The aforementioned methods of exploiting quantum correlations to establish quantum communication are based on quantum error correction procedures limiting the negative effects of the system's interaction with the environment. Meanwhile, quantum biology delivers examples of systems that benefit from the interaction by utilizing it as a catalyst in transport or measurement processes. The molecular light-harvesting complexes are an example of systems with noise-assisted transport mechanisms [9]. Crucial processes proposed to explain this behavior include noise-caused line broadening, environment-induced unitary evolution and phase decoherence leading to channel activation, or interaction between excited states of a molecular network with a continuous or discrete spectrum of vibrational modes [10]. The work [11] contains an analysis of a transport mechanism in quantum networks interacting with a spin bath. Coupling to a spin bath is used as well in magnetodetection processes based on free-radical mechanisms [12]. The second goal of this thesis is to propose procedures exploiting quantum correlation decay stemming from this mechanism in experimentally realizable systems. We consider the evolution of a system of quantum dot electron spin qubits in gallium arsenide (GaAs) in an external magnetic field, where hyperfine coupling with maximally mixed state of nuclei in a dot is the main mechanism of decoherence. In manuscript [C] we present a procedure based on singlet-fraction measurement in a Pauli-blockade regime. It aims at measuring the value of a magnetic field by exploiting its connection with the time of sudden death of entanglement in the system. The work [D] is devoted to the analysis of the evolution of general quantum correlations, and its utilization in magnetodetection.

In order to introduce the notation, below we present the basics of quantum-information and a physical description of the considered problems, starting from a sketch of quantum error correction techniques based on stabilizer codes.

The quantum threshold theorem [13] is concerned with the possibility of performing quantum computation in the presence of noise. It states that every quantum computation can be performed with arbitrary precision with a poli-logarithmic overhead in space and time, provided that the probability of a local error is below some threshold value (dependent on the quantum code architecture), and that the error correlations decay exponentially. A critical discussion of the local noise model, based on open system dynamics of quantum systems interacting with a thermal bath, is contained in the work [14]. In our analysis ([A], [B]) concerning quantum communication, due to the fact that the Hamiltonians of our error correcting codes contain only local interactions, and that in order to perform teleportation of a state of a quantum code we use noisy quantum channels acting separately on every qubit[1] of the code, we take local noise to model the interaction of every qubit state with the environment:

$$\rho \to (1-p^2)\rho + p(1-p)\sigma_x\rho\sigma_x + (1-p)^2\sigma_y\rho\sigma_y + p(1-p)\sigma_z\rho\sigma_z, \qquad (1)$$

where by $\sigma_x, \sigma_y, \sigma_z$ we denote the Pauli matrices, $0 \leq p \leq 1$.

The stabilizer formalism [15] allows for a convenient description of many quantum error correcting codes granting protection against decoherence in the sense of the quantum threshold theorem. Here, the quantum state is stored within a logical subspace $\mathcal{H}_{log}$ of a Hilbert space of a system of N physical qubits, $\mathcal{H}_{sys} = \otimes_i \mathcal{H}_i$, $i = 1, \ldots, N$: $\mathcal{H}_{log} \subset \mathcal{H}_{sys}$, with the decomposition into logical qubits $\mathcal{H}_{log} = \otimes_j H_{L,j}$, $j = 1, \ldots, D \leq N$ and space dimensions $dim[\mathcal{H}_i] = 2 = dim[\mathcal{H}_{L,i}]$. The logical subspace $\mathcal{H}_{log}$ is spanned by eigenvectors of the

[1] i.e. a quantum two-level system.

group of operators (called code stabilizers) S: $\{|\Psi\rangle : s|\Psi\rangle = |\Psi\rangle, \forall s \in S\}$, where S is an abelian subgroup of a Pauli group acting on $\mathcal{H}_{sys}$, that does not contain $-\mathcal{I}$, where $\mathcal{I}$ is the identity operator. The generator of S, $G(S)$, can be defined as the set of hermitian, pair-wise commuting operators from a Pauli group, with cardinality $|G(S)|$. The dimension of the logical subspace satisfies $dim[\mathcal{H}_{log}] = N - |G(S)|$. Elements of pairs of anti-commuting operators $(X_{L,j}, Z_{L,j})$ on $\mathcal{H}_{L,j}$ commute with all elements from $G(S)$, but cannot be created from $G(S)$. In the case of CSS codes, $G(S)$ can be represented as a set of operators in which every operator is a tensor product of one type of Pauli matrix (σ_x or σ_z) or the identity operator. In the works [A] and [B] we will consider topological stabilizer codes (i.e. defined on a network), with stabilizers defined locally on quantum systems of neighboring nodes. Measurements of the stabilizers enable the protection of quantum correlations against local noise; logical operators are defined in a non-trivial way on loops uncontractible to a point.

Below we discuss the case of modeling the interaction of quantum dots with an environment, as the dependence of the evolution of quantum correlations evolution on the magnetic field in these systems is the main topic of works [C] and [D]. A conduction band of gallium arsenide is constructed mainly from s-type orbitals. Therefore, isotropic hyperfine interaction dominates other terms (stemming from a solution of the Dirac equation for an electron): factors proportional to momentum (its coupling to nuclear spins, spin-orbit interaction) or containing an average over (spherically symmetric) electron wavefunction in the neighborhood of nuclei (non-isotropic hyperfine coupling) [16]. Due to the symmetry of charge distribution in the neighborhood of a nucleus, quadrupolar interaction in gallium arsenide (containing spin 3/2 nuclei) can be neglected as well. The time scales of the internal evolution of a spin bath are determined by dipole-dipole interaction H_{dip} in a secular approximation (valid for magnetic fields $B > 0.1\ mT$)

$$H_{dip} = \sum_{k\neq l} d_{kl} \hat{I}_k^z \hat{I}_l^z - 2 \sum_{k\neq l} d_{kl} \hat{I}_k^+ \hat{I}_l^- \tag{2}$$

between spins $\hat{I}$ of the nuclei, where $\hat{I}^{\pm} = \hat{I}^x \pm i\hat{I}^y$. Due to the weakness of dipole-dipole interactions (with a scaling $d_{kl} \propto 1/r_{kl}^3$, where r_{kl} is the distance between nuclei labeled by k and l), the effects of spectrum broadening (the first term of eq. (2)) and spin diffusion (the second term) are important in time-scales, respectively, of 100 μs and 1 s [17], which enables one to neglect H_{dip}

in further considerations of system evolutions in a nanosecond regime. Therefore, neglecting Zeeman splitting of nuclei ω_k, which are approximatively 10^3 times smaller than the respective electron splitting (assuming that the evolution time t satisfies $t \ll \min_{\omega_k,\omega_l} \frac{1}{|\omega_k - \omega_l|}$ for all nuclei k, l in a quantum dot), in publications [C], [D] we consider only the electron interaction with an external magnetic field and isomorphic hyperfine interaction (in the approximation of effective Hamiltonian for an electron in a ground state)

$$H = -g\mu_B B\hat{S}^z + \sum_k A_k \hat{S}\hat{I}_k, \tag{3}$$

where g is the effective giromagnetic ratio of the electron, μ_B the Bohr magneton, and $\hat{S}^z$ a component of a spin operator paralell to the magnetic field. Hyperfine couplings $A_k \propto A^k|\psi_0(\mathbf{r}_k)|^2$ depend on the value of the electron envelope wavefuntion at the site of the nucleus $\mathbf{r}_k$: $\phi_0(\mathbf{r}_k) = \sqrt{v_0}u(\mathbf{r}_k)\psi_0(\mathbf{r}_k)$, with a crystal elementary cell volume v_0 and $u(\mathbf{r})$ being a periodic Bloch function associated with a wave vector $\mathbf{k} = 0$ [18]; the material constant A^k depends on a chemical element and the isotope of the nuclei.

In work [C] quantum entanglement is measured by the concurrence function [19, 20]. For a two qubit state ρ this measure is defined as

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{4}$$

where λ_i are the square roots (in descending order) of the eigenvalues of a matrix $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where * denotes a complex conjugation. In the manuscript [D] we investigate the possibility of exploiting for magnetometric purposes the evolution of general quantum correlations in these systems. We employ a geometric measure $D_S(\rho)$ [21] of quantum discord [22] as an indicator of general quantum correlations. Despite of a lack of symmetry under an exchange of subsystems, this indicator serves as a fidelity measure for remote state preparation schemes [23, 24]; its evolution in the considered systems (and the evolution of associated observables) allows for suggesting magnetodetection protocols in magnetic field regimes not accesible for methods presented in [C]. In order to minimize the geometric discord dependence on purity of a state (which is a result of the non-contractivity of a Hilbert-Schmidt norm with respect to local completely positive trace preserving operations [25, 26]), we use a rescaled geometric discord [27]

$$D(\rho) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)\left[1 - \sqrt{1 - \frac{D_S(\rho)}{2Tr[\rho^2]}}\right]. \tag{5}$$

$D_S(\rho)$ is estimated by calculations of its bounds: an upper [21]

$$D_S^{'}(\rho) = \frac{1}{4}\max_{q=x,y}(Tr[K_q] - k_q), \tag{6}$$

and a lower one [28]

$$D_S^{''}(\rho) = \frac{1}{4}\max_{q=x,y}(Tr[K_q] - k_q + Tr[K_p] - k_p), \tag{7}$$

for $p \neq q$. Using a correlation matrix representation of a two qubit state, with matrix elements defined as $T_{ij} = Tr[\rho(\sigma_i \otimes \sigma_j)]$ and vectors $\vec{x} = |x\rangle$, where $x_i = Tr[\rho(\sigma_i \otimes \mathcal{I})]$ and $\vec{y} = |y\rangle$, where $y_i = Tr[\rho(\mathcal{I} \otimes \sigma_i)]$, we define k_q as the biggest eigenvalue of a matrix $K_q = |q\rangle\langle q| + T_q T_q^T$, with $T_x = T$, $T_y = T^T$. l_q is the biggest eigenvalue of a matrix $L_q = |q\rangle\langle q| + T_q|\hat{k}_p\rangle\langle\hat{k}_p|T_q^T$, while $|\hat{k}_q\rangle$ is a normalized eigenvector associated with the eigenvalue k_q of the matrix K_q. $D_S^{'}$ and $D_S^{''}$ converge e.g. for states diagonal in the Bell basis.

B. Summary of results contained in PhD thesis

Below we present a summary of publications that consist for the PhD dissertation. They are divided into two groups: works [A-B] are dedicated to the application of quantum error correcting codes in quantum communication and quantum memories; they contain analytic bounds for the fidelity of associated quantum processes in the presence of local noise, and take into account chosen architectures. The second group of publications, [C-D], is devoted to the interaction of realistic systems of two electron spin quantum dots in gallium arsenide with their separate spin environments. We present protocols aimed at the detection of external magnetic field that are partly achievable in experiment. They are based on a relation between the value of a magnetic field and the type and rate of decoherence. We begin with a presentation of proofs for the possibility of quantum communication in a two-dimensional quantum network.

1. *Long-distance quantum communication in two and three dimensions in presence of noise*

Publication [A] presents a proof of quantum communication in noisy two-dimensional networks, together with a simple scheme of communication in three dimensions. In both cases the reasoning is based on an isomorphism between quantum information storage in quantum networks of dimension D, and information transfer in a dimension $D+1$. Introduced in [29], the isomorphism relies on the fact that procedures which enable the correction of local noise resulting from interactions with environment in time, can be likewise applied in a situation when the errors stem from the teleportation of a code structure in space, from one layer to another, with the use of non-maximally entangled states shared between pairs of network nodes from different layers. Therefore, in order to prove the possibility of quantum communication in two and three dimensions, we calculate the fidelity of quantum memories in one and two dimensions, respectively. We take into account not only the noise effect on information storage, but also limited fidelities of encoding and decoding procedures.

In the one-dimensional case ([A], Proposition 1) it is shown that the fidelity F_0 of encoding of an unknown qubit state into a concatenated code is lower-bounded by a function

$$F_0 \geq e^{-2v\sqrt{p}}, \tag{8}$$

where v is the number of gates used for encoding into the first level of concatenation, $p \leq p_{th}/3$ is the probability of a local noise (acting on a single qubit after performing each of the gates, including identity gates), which is smaller than the threshold probability p_{th} of a chosen code architecture. The proof relies on the calculation of the success probability of encoding into the r-th level of concatenation, which exploits the estimation of the error probability p_k on the k-th level ($p_k \leq \frac{1}{c}\left(cp_0\right)^{2^k}$, with $c = \binom{v}{2}$) and algebraic estimations performed in order to relinquish the dependence on the parameter r. A reasoning included in the work [30] allows for calculating a lower bound on the fidelity of quantum state storage. We assume that the fidelity of decoding is not lower than the encoding fidelity (which is satisfied e.g. for a full unitary fault tolerant scheme [13]). On a basis of the aforementioned isomorphism between quantum memory and quantum communication, we receive a lower bound on the fidelity F of quantum communication in two dimensions:

$$F > e^{-2v\sqrt{p}}[1 - 2c_1 T e^{-c_2 V}], \tag{9}$$

where $V = N^k$ is the number of qubits and T the number of space steps on which the teleportation takes place. It is required that the communication scheme utilizes resources of local entanglement between nodes of a network, and that the local error does not exceed $p_{th}/3$. Constants c_1 and c_2 depend on the type of concatenated code. The bound (9) implies a possibility of entanglement percolation [29] in a two-dimensional quantum network of nodes sharing entanglement with their closest neighbors.

We start our analysis of quantum communication in three-dimensional quantum networks from the non-fault tolerant case, where we assume the following: it is possible to ideally prepare a qubit state in an eigenstate of σ_z and σ_x Pauli matrices (i.e. $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$, $\sigma_x|\pm\rangle = \pm|\pm\rangle$); all the qubits can be measured in the σ_z and σ_x Pauli bases with no fault, as well as the values of all the stabilizers; one can perform σ_x and σ_z rotations on the code qubits ([A], part III). We present encoding and decoding procedures of an unknown qubit state into a topological Kitaev code on a surface. State preparation is depicted on Fig. 2(a) ([A]). Succeeding stabilizer measurements and qubit rotations are aimed at driving the state of a code into the logical subspace. The decoding procedure ([A], Rys. 2(b)) is based on rotations performed on the qubit on which the code state is to be decoded. Those operations are conditioned on the one-qubit measurements in the Pauli basis, which destroy correlations in the code. Correctness proofs of the aforementioned procedures ([A], Proposition 2, Proposition 3) are based on an observation [31] that the fidelity of a quantum process depends only on the outcomes of the measurements performed on two complementary sets of input states. Part IIIB of the manuscript [A] contains a description of the encoding procedure viewed as a teleportation scheme of an unknown state from one node of a network into a two-dimensional quantum code. Maximally entangled virtual qubits, created in a code by the stabilizer measurements, are the resource used in this teleportation process.

We move to the fault-tolerant scenario, where state preparation and measurements of stabilizers are subjected to noise ([A], part IV). In order to detect and eliminate these errors, we extend the non-fault tolerant encoding and decoding procedures by stabilizer measurements performed in time, as well as by a proper error correction algorithm. We calculate a lower bound on the fidelity of encoding, storage and decoding of an unknown qubit state in a Kitaev planar code. While assuming a local noise model, we do not take into account error propagation to quantum codes from the measurements of the stabilizers values. Nevertheless, we believe that this issue can be addressed in the spirit of work

[32], where the application of proper schemes results in a modification of the threshold probability value enabling quantum calculations with the arbitrary precision, however, it do not affect the positivity of the threshold value.

A scheme for correcting phase errors (bit errors can be corrected independently and in an analogous way due to the CSS structure of the code) relies on performing chains of σ_z rotations on two-dimensional projections of the paths connecting those of the stabilizers X_s in a three-dimensional network that registered non-trivial measurement outcomes – it indicates that the support of the code state in not contained in the logical subspace ([A], Fig. 4). The paths are set so that their total length in a cubic network with Taxicab metric is minimized - the metric is modified by a weight factor $-\ln \frac{p_i}{1-p_i}$ on the edges that represent qubits or stabilizer measurements that are affected by local noise with probability p_i. This modification allows to take into account the effects of encoding and decoding procedures, as those procedures are based on the preparation of qubit states and their measurements in a chosen basis. As a result, there is a very high probability of obtaining a non-trivial measurement outcome of a stabilizer of a type dual to that of the basis (e.g., measurement of X_S stabilizer defined on qubits all prepared in the σ_Z basis). Therefore, from the perspective of the procedures which correct errors of a chosen type, the effective boundaries on the preparation and measurement code layers experience a shift towards a diagonal of the two-dimensional cuts ([A], Fig. 4). This illustrates that it is not possible to avoid an impact of the encoding and decoding parts of the protocol onto the fidelity of the communication process.

This intuition is rigorously justified by analytic estimates of the presented procedure fidelity ([A], Proposition 6). They are based on estimating the probability of occurrence of non-trivial error loops that lead to an uncontrolled logical operation on an unknown code state. Independent calculations for bit and phase errors ([A], eq. (17) and (18)), lead, through Lemma 4 and under an assumption of at most polynomial increase of time storage with code dimension N, to the following bound of the process fidelity

$$\lim_{N\to\infty} \overline{F} \geq 1 - 6p - \frac{2\alpha^2(5-3\alpha)}{(1-\alpha)^3} \tag{10}$$

for $p \lessapprox 0.007$, with $\alpha = 12\sqrt{p(1-p)}$.

Due to the isomorphism between the time and space dimension of the scheme, the above expression applies to the fidelity of the protocol of quantum communication in three dimensions, relying on *local, short-term* memory and *local* resources of entanglement ([A], Fig. 8). Implication 2 shows that the space dimensions of a code scale at most logarithmically with the distance of quantum communication. Therefore, the described procedure is a protocol of entanglement percolation. It requires commuting measurements of stabilizers on code layers, transversal teleportation of code structures between the layers, as well as qubit preparation and measurements aimed at encoding and decoding of an unknown quantum state.

2. *Encoding of an unknown quantum state into CSS code*

Work [B] is aimed at generalizing the procedure of encoding an unknown qubit state into a Kitaev planar code to all CSS quantum error correcting codes, which was introduced in [A]. We define the encoding of a state $|\Psi\rangle_i = \alpha_i|0\rangle_i + \beta_i|1\rangle_i$ of a physical qubit on Hilbert space $\mathcal{H}_i$ as a process under which the logical qubit, defined on subsystem j of the logical subspace $\mathcal{H}_{log} = \otimes_j \mathcal{H}_{L_j}$, is transferred to a state $|\Psi\rangle_{L_j} = \alpha_i|0\rangle_{L_j} + \beta_i|1\rangle_{L_j}$. The proposed encoding scheme relies on the property of CSS codes, that all logical operators are the tensor product of only one Pauli matrix type (σ_z or σ_x). The proposed non-fault tolerant procedure applies to an arbitrary CSS code – nevertheless, due to the simplicity of its generalization into a fault-tolerant regime for a topological architecture, we restrict ourselves to this kind of code constructions.

The non-fault tolerant scheme of encoding un unknown qubit state into a topological CSS code requires the preparation of states $|0\rangle$, $|+\rangle$ on the qubits where the logical operators Z_L and X_L, respectively, are non-trivially defined. Each of the logical operators acts non-trivially on qubits situated on curves that connect code boundaries. Those curves cross at an odd number of qubits. Only a qubit in a state to be encoded is put at one of these crossing points, whereas pairs of qubits from different crossings (if they are present) are prepared in maximally entangled states. Other qubits (those, on which the logical operators do not act) can be prepared in an arbitrary state. Stabilizer measurements and error correction bring the state of the system to the logical subspace. The proof of correctness of the scheme relies on the fact that the joint parity of the qubits situated on the logical operator curves, originally dependent only on the qubit state to be encoded, remains so under the encoding procedure. This is achieved by applying a logical operator dual to the logical operator that was crossed an even number of times by non-commuting correction paths. Nevertheless, in part III we show that for Kitaev code on a torus [33], Kitaev code with defects [34, 35], Bravyi sybsystem code [36] and Haah code [37] it is possible to perform corrections in a way that they do not cross logical operator curves. Therefore, applying logical operations to the code can be eliminated.

For many CSS codes, an extension of the above reasoning to the fault-tolerant scenario is based on preparing qubits that surround curves of logical operators Z_L and X_L in states $|0\rangle$ and $|+\rangle$, respectively, and on the error correction scheme based on a history of stabilizer measurements, similarly as in work [A]. The manuscript [B] contains a description of the decoding of an unknown qubit state from CSS codes. In part IV we calculate an upper bound on the phase error probability of a quantum memory based on the proposed algorithms applied to Bravyi subsystem code. We show that, in the limit of the infinite code size, the fidelity of the quantum memory is of the order $1 - \mathcal{O}(p)$, where p is the probability of local noise acting on qubits during one time step, as well as the classical error probability of a stabilizer measurement.

It is worth noticing that, by enabling magical state encoding [8], the above procedures can lead to the realization of universal quantum computation on CSS codes.

3. *Entanglement decay in systems of quantum dot spin qubits in the presence of the magnetic field*

In manuscript [C] we consider a connection between the value of sn external magnetic field and the character and type of entanglement decay in a system of two mutually noninteracting electron spin quantum dots in gallium arsenide. Dipole-dipole interaction between nuclei in each of the dots, as well as their Zeeman interaction with the field are negligible for short evolution times in which the entanglement decay occurs. The hyperfine interaction between electron spin and an environment of nuclear spins in maximally mixed state constitutes the main mechanism of decoherence, and the system Hamiltonian takes the form $H = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2$, with single quantum dot Hamiltonians (3)

$$H_i = \Omega \hat{S}_i^z + \hat{D}_i + \hat{V}_i, \tag{11}$$

where $\Omega = -g\mu_B B$ is the Zeeman splitting. The $\hat{D}_i$ and $\hat{V}_i$ operators describe the hyperfine interaction between the spin ($\hat{\mathbf{S}}_i$) of an electron labeled by an index i, and spins ($\hat{\mathbf{I}}_{k,i}$) of the nuclei labeled by k, with coupling constants $A_{k,i}$. The term $\hat{D}_i = \sum_k A_{k,i}\hat{S}_i^z\hat{I}_{k,i}^z$ leads to dephasing, while $\hat{V}_i = \frac{1}{2}\sum_k A_{k,i}(\hat{S}_i^+\hat{I}_{k,i}^- + \hat{S}_i^-\hat{I}_{k,i}^+)$ results in dephasing and a change of occupation levels, using the standard notation $\hat{S}_i^\pm = \hat{S}_i^x \pm \hat{S}_i^y$ and $\hat{I}_i^\pm = \hat{I}_i^x \pm \hat{I}_i^y$.

The constants $A_{k,i}$ depend on nucleui positions with respect to the electron wavefuntion. As the number of nuclei in a realistic quantum dot in gallium arsenide is of the order of $N \approx 10^6$, it is not possible to rigorously solve the system evolution analytically nor numerically. Due to the slow internal dynamics of the spin bath resulting from the weakness of dipole-dipole interactions, the Markov approximation is not applicable. In the interesting regime of weak magnetic fields ($\Omega \ll A = \sum_k A_{k,i}$), it is not possible to use a weak coupling approximation. Nevertheless, we deduce from the energy-time uncertainty relation [17] that in the time regime $t \ll \frac{N}{A} \approx 10\ \mu s$ the exact distribution of the $A_{k,i}$ constants in gallium arsenide should be irrelevant. Therefore, we apply the commonly used model [38, 39] and take $A_{k,i} = A/N$ in both dots. As a result, in the total angular momentum basis of the nuclei we obtain a block form of a single dot Hamiltonian. It allows its diagonalization. In the appendix ([C]) we showed that coherence revivals stemming from the interaction with a small number of nuclei are strongly suppressed with increasing N. As a consequence, the evolution of a single dot can be effectively simulated by the interaction with only 50 nuclei. In the regime of high magnetic fields ($\Omega \gg A$), the V_i term can be omitted (due to the conservation of energy in processes involving changes of the electron spin occupation levels), and decoherence is limited to pure dephasing, with coherences decaying proportionally to $e^{-t^2/T_2^{*2}}$, with characteristic time $T_2^* \propto \sqrt{N}/A$. For smaller magnetic fields, oscillations of coherences remain correlated with oscillations of occupation levels. For very small magnetic fields ($\Omega \lesssim A/\sqrt{N}$) the occupation levels become partially evened.

We employed the concurrence function (4) as a measure of entanglement between two noninteracting quantum dots. Due to the impact of the magnetic field value onto occupation levels of electron spin system, by modifying the value of this external parameter of evolution one can change the character of entanglement decay: from a Gaussian type to the one exhibiting sudden death of entanglement ([C], Fig. 1). The time of sudden death t_{SD} increases non-monotonically with the increment of the magnetic field B ([C], Fig. 2), displaying a logarithmic dependence in the high magnetic field limit. We analyze the possibility of detecting small magnetic field values by using the dependence of the time of sudden death with a magnetic field. Inclined by an experimental realization of a measurement of singlet state fidelity F in a double quantum dot system (in a Pauli blockade regime [40]), we introduce an entanglement witness $W(t) = \frac{1}{2} - F(t)$ that satisfies the relations $t < t_{SD} \implies W(t) < 0$ and $t = t_{SD} \implies W(t) = 0$, which stems from the bistochastic character of the quantum channels in both dots. We suggest that measurements of $W(t)$ be used in the regime of step dependence of $t_{SD}(B)$ as a threshold sensor of the magnetic field.

4. *Evolution of quantum correlations and protocols for magnetic field detection in systems of quantum dot spin qubits*

Work [D] is a continuation of research aimed at exploiting quantum noise for magnetic field detection in quantum dot systems considered in [C]. Having set the question whether entanglement is a necessary resource for magnetic field detection (in the sense of work [C]), we investigate decay of quantum correlations described by a rescaled geometric discord D (5). In comparison with decay of entanglement, the decay of discord of Bell states is characterized by the lack of oscillations visible in the entanglement evolution, as well as by a revival for long evolution times, for small values of the magnetic field ([D], Fig. 1). The revival can be used for magnetometric purposes in a range $0 - 5\ mT$ - it is facilitated by the fact that its measurement is identical with the measurement of the coherence in the system. The Change of the sign of the function $1 - g(t)$ at time t_0, $g(t) = \frac{Tr(\sigma_z \otimes \sigma_z \varrho(t))}{Tr(\sigma_x \otimes \sigma_x \varrho(t))}$, is a necessary and sufficient condition for discord non-differentiability at the time t_0. For zero magnetic field, transitions from a singlet to all triplet states are equally probable, which implies $g(t) = 1$ for arbitrary t. For non-zero magnetic fields, decoherence into triplet states is suppressed (due to the necessity of distributing the energy coming from electron spin change to the environment). This leads to $g(t) \leq 1$ for arbitrary time t. In this way, the energy conservation principle prevents non-differentiability of a discord function to occur during the evolution of a singlet state.

Werner states are defined by

$$\rho = (1-p)\mathcal{I} + pS_0, p \leq 1, \tag{12}$$

where $\mathcal{I}$ is the identity operator acting on the Hilbert space of a two qubit system, and $S_0 = \frac{1}{2}\Big(|01\rangle - |10\rangle\Big)\Big(\langle 01| - \langle 10|\Big)$ is the projector on the singlet state. Evolution of all Werner states (including separable (i.e. non-entangled) ones for $0 \leq p \leq 1/3$ parameter values) show a qualitatively similar dependence on the magnetic field. Fig. 3(a) ([D]) depicts the impact of the magnetic field onto the presence of quantum correlations in the evolution of a separable state. The presence is quantified by

$$M(B) = \frac{1}{D(\rho(0))}\int_0^{\tau} D(\rho(t))dt, \tag{13}$$

with $\tau = 20\ ns$. Therefore, entanglement is not a necessary resource in the magnetometric application of the electron spin qubits double quantum dot systems.

Another possible measurement method, applicable in the regime of weak magnetic fields, can be based on the non-directly measurable function $g(t)$. $g(t)$ exhibits a very strong dependence on the magnetic field B in a regime $0-2\ mT$ ([D], Fig. 2). As in the previous scheme, entanglement of the initial states is not required – this comes from the fact that all Kraus operators characterizing a single dot quantum channel either commute or anti-commute with single qubit rotations. For the same reason a non-ideal state preparation $\rho = (1-p)S_0 + pT_0$, where $T_0 = \frac{1}{2}\Big(|01\rangle + |10\rangle\Big)\Big(\langle 01| + \langle 10|\Big)$ is a projector on a triplet state (that can be distinguished from a singlet in a Pauli blockade scenario), results in a rescaling $g(t) \rightarrow \frac{1}{1-2p}g(t)$. For small p it does not significantly influence the sensitivity of the parameter under consideration to the magnetic field.

Moreover, we conclude that the system can display non-differentiability in the evolution of quantum discord as long as an initial state is chosen properly ([D], Fig. 3). We show that the discord evolution of non-diagonal Bell states depends strongly on non-local phase factors ([D], Fig. 4) – which remains in contrast with the evolution of entanglement.

C. Perspectives

We have summarized the results of our twofold approach to exploiting quantum correlations in the presence of external noise. Below we describe possible directions for future research in the two selected domains.

As the procedures introduced in publications [A] and [B] can be applied in universal quantum computing protocols, one can examine the fidelity of encoding a magical state into the considered code structures. One should take into account distillation procedures and methods aimed at minimizing the propagation of the measurement errors into a code. Another goal would be to extend the encoding fault-tolerant scheme into CSS topological codes defined on lattices with higher dimensionality, which is motivated by their postulated applications as self-correcting quantum memories. Tighter bounds on process fidelities would serve as an indicator of the purposefulness of practical implementation of the proposed procedures.

Studies of magnetic-field-influenced decoherence in electron spin double quantum dot systems, presented in publications [C] and [D], can be supplemented by the analysis of the impact on the decay of correlations that can be exercised by the exchange interaction between electron spins, as well as by the merging of environments of quantum dots. Both the effects appear for small distances between the dots. Magnetometric properties of the considered systems can be analyzed in terms of bounds for Fisher information of m independent quantum channels [41], its scalling with m, as well as by the optimization of a single measurement time with m [42]. Moreover, one can perform a search for an observable saturating the bound on the precision of magnetic field measurements.

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