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Reduced state of the field and classicality of symplectic time evolution

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Abstract

Despite more than a hundred years since the inception of quantum mechanics, there is still much to be said about the exact nature of its transition to classical physics. The problem has perhaps never been as relevant as today, with more and more experimental setups comprised of few-to-many body systems, which lie precisely in the intersection between classical and quantum. Along with these practical developments came new theoretical tools for their description, most notably, *mesoscopic* frameworks – self-contained formalisms streamlining the full quantum description to be simple, yet useful in the relevant regime of particle numbers.

Recently, one such mesoscopic framework came in the form of the *reduced state of the field (RSF)*, originally devised by Robert Alicki as a way to describe quantum features of macroscopic bosonic fields, e.g. light waves. In this dissertation, we show that RSF also serves as a viable tool for probing classicality within quantum mechanics, and subsequently use it to investigate the classicality of *symplectic time evolution*, responsible for modeling the vast majority of dynamics in contemporary quantum-optical experiments. We fulfill our goals through a series of three papers.

First, we prepare symplectic evolution for its future study of classicality by addressing a conceptual gap within. As we observe based on RSF, viewed from the perspective of its usefulness in modern quantum optics, symplectic evolution is incomplete, failing to describe certain operations available in experiment. In our First Paper, we derive the missing component and provide its in-depth physical interpretation, most notably as a description of random scattering and a tool for entanglement creation in dissipative engineering.

We then advance on our main goal in our Second Paper. After analyzing various aspects of RSF to prove its semiclassicality, most significantly demonstrating its very limited use as a description of entanglement, we employ it to derive the exact conditions for classicality of symplectic evolution. As we find, such evolution is semiclassical only if it consists of passive transformations, like beam-splitters, which have a natural interpretation within classical physics.

Finally, in our Third Paper, we supply our previous results by revisiting the classicality of symplectic evolution from the point of view of Bogoliubov transformations: linear transformations of the field's creation and annihilation operators used in nearly all branches of many-body quantum physics. In particular, we study the dynamical Casimir effect, finding that from the perspective of an open quantum system, the effect reduces to semiclassical dissipation.

Abstrakt

Pomimo ponad stulecia od narodzin mechaniki kwantowej, wciąż wiele pozostaje do powiedzenia w kwestii jej przejścia do fizyki klasycznej. Temat ten nie był być może nigdy tak ważny jak dziś, gdy coraz więcej doświadczeń wykonuje się z udziałem kilku-do-wielociałowych układów kwantowych, leżących na granicy między fizyką klasyczną i kwantową. Wraz z tym postępem pojawiły się też nowe narzędzia teoretyczne do jego opisu, w szczególności teorie mezoskopowe – formalizmy upraszczające mechanikę kwantową bez straty dokładności na omawianych skalach liczby cząstek.

Jedną z najnowszych takich teorii jest *zredukowany stan pola (RSF)*, skonstruowany oryginalnie przez Roberta Alickiego z myślą o opisie kwantowych cech makroskopowych pól, takich jak fale świetlne. W niniejszej pracy doktorskiej pokazujemy, że RSF służy także jako skuteczne narzędzie do badania klasyczności w mechanice kwantowej, a następnie stosujemy je do *ewolucji symplektycznej*, modelu ewolucji czasowej dominującego we współczesnej optyce kwantowej. Nasz cel osiągamy w serii trzech artykułów.

Zaczynamy od przygotowania ewolucji symplektycznej do przyszłych badań jej klasyczności. Jak zauważamy dzięki RSF, patrząc na ewolucję symplektyczną z perspektywy jej użyteczności w optyce kwantowej, należy uznać ją za niepełną, nie jest bowiem w stanie uchwycić wszystkich operacji dostępnych doświadczalnie. W Artykule Pierwszym wyprowadzamy dokładną postać brakującego elementu ewolucji i przedstawiamy jego szczegółową intepretację, z naciskiem na jego użyteczność w opisie zjawiska losowego rozpraszania i tworzeniu splątania w inżynierii dysypatywnej.

Następnie, w Artykule Drugim, wracamy do naszego celu głównego. Udowodniwszy półklasyczny charakter RSF, m. in. poprzez pokazanie znacznych ograniczeń w opisie splątania w formalizmie, używamy go do wyprowadzenia ścisłych warunków na klasyczność ewolucji symplektycznej. Jak pokazujemy, ewolucja taka jest półklasyczna tylko gdy składa się z tak zwanych pasywnych operacji, takich jak dzielniki wiązek, które posiadają naturalną interpretację w fizyce klasycznej.

Na koniec, w Artykule Trzecim, uzupełniamy otrzymane do tej pory wyniki drugim spojrzeniem na klasyczność ewolucji symplektycznej, tym razem z punktu widzenia transformacji Bogoliubowa: liniowych funkcji operatorów kreacji i anihilacji pola używanych w niemal wszystkich gałęziach fizyki kwantowej wielu ciał. Za główny przykład takich transformacji obieramy dynamiczny efekt Casimira, pokazując, że z perspektywy kwantowych układów otwartych, efekt ten redukuje się do półklasycznego procesu dysypacji.

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I would like to dedicate this dissertation to the memory of my kind aunt Irena Mierzejewska, as well as to Zofia Baczewska, who were always very supportive of my PhD.

Publications included in the dissertation

[First Paper] T. Linowski, A. Teretenkov, Ł. Rudnicki, *Dissipative evolution of quantum Gaussian states*, Phys. Rev. A **106**, 052206 (2022).

[Second Paper] T. Linowski, Ł. Rudnicki, *Reduced state of the field and classicality of quantum Gaussian evolution*, Phys. Rev. A **106**, 062204 (2022).

[Third Paper] T. Linowski, Ł. Rudnicki, *Classicality of the Bogoliubov transformations and the dynamical Casimir effect through the reduced state of the field*, Acta Phys. Pol. A **143**, S95 (2023).

Other publications

[OP1] K. Schlichtholz, T. Linowski, M. Walschaers, N. Treps, Ł. Rudnicki, G. Sorelli, *Practical tests for sub-Rayleigh source discriminations with imperfect demultiplexers*, arXiv:2303.02654 (2023).

[OP2] T. Linowski, K. Schlichtholz, G. Sorelli, M. Gessner, M. Walschaers, N. Treps, Ł. Rudnicki, *Application range of crosstalk-affected spatial demultiplexing for resolving separations between unbalanced sources*, arXiv:2211.09157 (2022).

[OP3] T. Linowski, Ł. Rudnicki, C. Gneiting, *Spectral stabilizability*, Phys. Rev. A **107**, 042218 (2023).

[OP4] T. Linowski, K. Schlichtholz, Ł. Rudnicki, *Formal relation between Pegg-Barnett and Paul quantum phase frameworks*, Phys. Rev. A **107**, 033707 (2023).

[OP5] T. Linowski, C. Gneiting, Ł. Rudnicki, *Stabilizing entanglement in two-mode Gaussian states*, Phys. Rev. A **102**, 042405 (2020).

[OP6] T. Linowski, G. Rajchel-Mieldzioć, K. Życzkowski, *Entangling power of multipartite unitary gates*, J. Phys. A: Math. Theor. **53**, 125303 (2020).

[OP7] A. Mandarino, T. Linowski, K. Życzkowski, *Bipartite unitary gates and billiard dynamics in the Weyl chamber*, Phys. Rev. A **98**, 012335 (2018).

Contents

Chapter 1

Introduction

Between the physics of a single particle, given by the wave function, and the high particle number regime, successfully described by (often classical) statistical methods, is the world of few-to-many-body systems. Today, the technological advancements allow for experiments with trapped ions [1, 2], cold atoms [3, 4, 5] and photons [6, 7] that lie precisely in this intersection between micro- and macroscopic ensembles (or in its vicinity). In such situations, on the one hand, the standard description in terms of the wave function or its second-quantized counterpart may prove too complex, often superseded by numerical simulations relying on, e.g. the Hartree-Fock method or the density functional theory [8]. On the other hand, statistical description breaks down for such, relatively small, particle numbers.

A proper treatment requires a *mesoscopic* theory [9]: an intermediate formalism between quantum and classical, which takes into account only features of the system relevant for the considered problem. Besides its value in simplifying the theory without damaging its practical accuracy, mesoscopic treatment has at least one more, conceptual advantage. Because it is neither quantum nor classical, comparing the mesoscopic description of a given phenomenon with its full quantum or classical treatment can lead to valuable insights into the phenomenon's degree of classicality, and in turn, help solve the decades old problem of the quantum-to-classical transition [10, 11]. Such conceptual considerations can even lead to further practical gain, as classical description is typically much simpler than quantum theory [12, 13, 14].

One area of physics where the problem of classicality is particularly vivid is optics. Classicality of light has been debated for a long time, beginning as early as in 1905 with Einstein's explanation of the photoelectric effect [15] and the discovery of wave-particle duality [16]. Despite progress, e.g. today it is generally accepted that Glauber's coherent states [17] with large amplitudes are essentially classical [18], the issue remains largely unresolved, even when considering a single photon [19, 20]. Similar considerations concern time evolution: both classical and quantum electromagnetic fields evolve under the same set of four Maxwell's equations [21, 22].

In modern quantum optical experiments, the majority of generated states are Gaussian, i.e. given by Gaussian characteristic functions and quasiprobability distributions [23, 24, 22]. The corresponding operations consist primarily of Gaus-

sian transformations and measurements [25, 26] and give rise to time evolution described by the Gorini–Kossakowski–Lindblad-Sudarshan (GKLS) equation generated by a polynomial of at most second degree in the quadrature operators. Over the years, such a setting has featured prominently in, e.g. quantum information processing [27, 28, 29], as well as works on entanglement [30, 31], discord [32], purity [33, 34], fidelity [35], steering [36], decoherence [37], and even quantum thermodynamics [38, 39], among others. Such Gaussian setup is most naturally described in the so-called *symplectic picture* – a formalism built completely upon the mean values of polynomials of up to second degree in the field's creation and annihilation operators. Crucially, while the symplectic picture is most closely associated with Gaussian states of light, for which it is fully equivalent to the standard quantum formalism [40], it can be also applied to non-Gaussian phenomena, in which case the framework becomes mesoscopic. If so, it can be employed to probe their classicality.

To this day, classicality of symplectic evolution, i.e. time evolution in the symplectic picture, and, more broadly, quantum evolution of light and Gaussian wavepackets, was investigated using a number of methods: relying on phase-space and the Winger distribution, hybridization of quantum and classical theories, and path integrals [41, 42, 43, 44]. The main goal of our dissertation is to approach the problem of classicality of light in the symplectic picture using another, more recent mesoscopic formalism of the *reduced state of the field (RSF)* [45]. Originally, RSF was designed to capture the quantum features of macroscopic, and therefore typically semiclassical, fields of a single particle type, including light fields. In particular, the formalism was successfully applied to polarization optics, bridging the Mueller and Jones calculi. The leading idea behind our thesis is to turn the original interpretation behind RSF on its head and show that, complementarily to its original goal, RSF also captures the semiclassical aspects of quantum fields. If so, it can be used to derive the exact conditions under which symplectic evolution, and thus all the phenomena described by it, should be considered classical.

Our ambitions are achieved over the course of three papers.

Our First Paper does not address our central problems directly, but serves as an important building block towards it: it fills in a blank in symplectic evolution itself, in this way making its subsequent study of classicality more complete. As discussed above, the symplectic picture, including its evolution, is designed to describe Gaussian states, due to the latter's relative accessibility in contemporary experimental setups. However, according to recent developments in quantum resource theories of non-Gaussianity [25, 26], Gaussian states and their convex combinations, which together form the set of so-called *quantum Gaussian states*, are equally easy to generate and manipulate. From this perspective, symplectic evolution should be restricted not to Gaussian states, but to the more general quantum Gaussian states. This is exactly what our First Paper addresses: following the time evolution in RSF [45], we choose the Lindblad operators defining the GKLS equation to be unitary, and show that the resulting evolution fits in with the symplectic picture. In addition, we investigate its physical interpretation, most notably regarding random scattering, as well as in dissipator engineering.

With this missing piece found, our Second Paper advances on our main goal of identifying the symplectic evolution's classicality. We do it in two steps. First, we investigate the RSF formalism itself to show that it indeed forms a semiclassi-

cal framework: we prove that it contains at most very limited information about quantum entanglement, and that its entropy behaves semiclassically. Then, we compare quantum Gaussian evolution with the time evolution model associated with RSF, deriving in this way the semiclassical subset of quantum Gaussian evolution. The classicality of the obtained evolution is intuitive: it allows exclusively passive transformations, which correspond solely to experimental operations that are energy-preserving and can be understood by treating light as a classical wave, such as beam-splitters and phase-shifters. This, in particular, applies to the missing scattering component found in the First Paper, which can be semiclassical only if the scattering matrices constituting it are passive.

Finally, for our Third Paper, we again look at classicality of symplectic evolution, but this time from the perspective of *Bogoliubov transformations* – linear transformations of the field's creation and annihilation operators preserving the canonical commutation relations [46]. Although they originated in studies on superconductivity [47, 48], today Bogoliubov transformations are considered an indispensable tool in a much wider area of physics, including optics, magnetism and even quantum field theory (including Unruh effect and Hawking radiation) [49, 50, 51]. Importantly, just like quantum Gaussian evolution, they are described by symplectic operations. In our Third Paper, we derive the exact conditions for classicality of Bogoliubov transformations, and apply them to the particular case of the *dynamical Casimir effect* – spontaneous production of particles in a medium following from non-trivial time dependence of either its boundary or its material coefficients [52, 53, 54, 55]. Curiously, we find that although the phenomenon is typically considered to be exclusive to quantum field theory, it can also be seen as a semiclassical dissipative effect.

This dissertation is organized as follows. In Chapter 2, we describe the necessary preliminaries: symplectic picture, Bogoliubov transformations (including the dynamical Casimir effect), as well as RSF. In Chapter 3, we summarize the main results of the papers included in the dissertation. Finally, in Chapter 4, we conclude and provide outlooks. Our papers are provided as attachments at the end of the dissertation.

Notational remark. In this work, we employ three different formalisms for quantum mechanics. For clarity, we use different notation for operators in each of them. Operators associated with the standard, density operators picture are denoted by "hats", e.g., $\hat{\rho}$. Operators associated with the symplectic picture are denoted by capital letters with no "hats", e.g., V. Operators associated with the RSF framework are denoted by small letters, also with no "hats", e.g., r.

Chapter 2

Preliminaries

2.1 Symplectic picture

Let us consider an N-mode, continuous variable Hilbert space $\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k$ equipped with N pairs of hermitian quadrature operators $\hat{x}_k, \hat{p}_k$ obeying the canonical commutation relations (we set $\hbar = 1$)

$$[\hat{x}_k, \hat{x}_{k'}] = [\hat{p}_k, \hat{p}_{k'}] = 0, \quad [\hat{x}_k, \hat{p}_{k'}] = i\delta_{kk'}. \tag{2.1}$$

Here and below, $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denote the ordinary commutator and anticommutator, respectively. As the quadrature operators form a basis of operators acting on $\mathcal{H}$, every density operator $\hat{\rho}$ describing the system can be fully characterized [56] by the complete $(n = 1, \ldots, \infty)$ set of *n-th order correlation functions*, or *n-th moments*, of the form

$$\langle \hat{\xi}_{l_1} \ldots \hat{\xi}_{l_n} \rangle_{\hat{\rho}} := \mathrm{Tr} \left[\hat{\rho} \, \hat{\xi}_{l_1} \ldots \hat{\xi}_{l_n} \right], \tag{2.2}$$

where we conveniently collected the quadratures operators in a single vector

$$\hat{\vec{\xi}} := \left(\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_N, \hat{p}_N \right)^T. \tag{2.3}$$

In many scenarios, especially those involving Gaussian states, i.e. states with Gaussian characteristic functions [23, 24, 22], sufficient information about the system is contained within the first and second moments only. In contrast to the infinitely-dimensional density operator, the first two moments are completely described by a moderate number of degrees of freedom, making their analysis much easier in comparison. In the arising *symplectic picture*, instead of by the density operator, the state of the system is described by the pair $(V, \vec{\xi})$. The $2N$-dimensional *vector of means*

$$\xi_k := \langle \hat{\xi}_k \rangle_{\hat{\rho}} \tag{2.4}$$

contains all the first moments, characterizing local information about the modes. The $2N \times 2N$ *covariance matrix*

$$V_{kk'} := \frac{1}{2} \langle \{ \hat{\xi}_k, \hat{\xi}_{k'} \} \rangle_{\hat{\rho}} - \xi_k \xi_{k'} \tag{2.5}$$

contains all the second moments, including non-local correlations between the modes, such as quantum entanglement. Taking into account the fact that some of the second moments are co-dependent [due to the canonical commutation relations (2.1)], it is easy to see that an N-mode symplectic state $(V, \vec{\xi})$ is described by $N(2N + 3)$ independent parameters, much fewer than the potentially infinite number of parameters defining a generic density operator.

Crucially, all the standard notions known from the density operator picture translate in a natural way to the symplectic picture. The canonical commutation relations (2.1) are encoded in the *symplectic form*

$$J_{kk'} := -i\big[\hat{\xi}_k, \hat{\xi}_{k'}\big], \tag{2.6}$$

explicitly equal to

$$J = \bigoplus_{k=1}^{N} J_2, \quad J_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.7}$$

The Heisenberg uncertainty relations

$$\sqrt{\langle \hat{x}_k^2 \rangle_{\hat{\rho}} - \langle \hat{x}_k \rangle_{\hat{\rho}}^2}\sqrt{\langle \hat{p}_k^2 \rangle_{\hat{\rho}} - \langle \hat{p}_k \rangle_{\hat{\rho}}^2} \geqslant \frac{1}{2}, \tag{2.8}$$

are equivalent to [23]

$$V + \frac{i}{2}J \geqslant 0. \tag{2.9}$$

Unitary operations $\hat{\rho} \rightarrow \hat{U}\hat{\rho}\hat{U}^\dagger$ on the density operator give rise to *symplectic operations* $(V, \vec{\xi}) \rightarrow (KVK^T, K\vec{\xi})$, defined as matrices preserving the symplectic form:[1]

$$KJK^T = J. \tag{2.10}$$

Finally, just like any density operator can be diagonalized by a unitary operation and is therefore described by its eigenvalues, any covariance matrix can be brought to a diagonal form by a symplectic operation and is described by its *symplectic eigenvalues*:

$$1/2 \leqslant \nu_1 \leqslant \ldots \leqslant \nu_N. \tag{2.11}$$

The symplectic eigenvalues come in pairs, i.e. the diagonalized covariance matrix reads $V_{\text{diag}} = \text{diag}(\nu_1, \nu_1, \ldots, \nu_N, \nu_N)$.

The symplectic picture is even sufficient to certify the presence of entanglement in the system, or more precisely, *distillable entanglement*, which is the only form of entanglement that can be used for practical tasks, such as quantum code encryption or teleportation [57]. According to the *positive partial transpose (PPT) criterion* for continuous variable systems [58, 59], if the partial transposition of the

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[1]Some define symplectic matrices by an alternative relation $K^T J K = J$. However, both definitions are completely equivalent, since when K is symplectic, so is K^T. Indeed, without loss of generality, consider $KJK^T = J$. Then $K^T = JK^{-1}J^{-1}$, which substituted into $K^T J K$ yields J. Proof in the other direction is analogous.

state with respect to a given bipartition is not positive semidefinite, then the state contains distillable entanglement with respect to this bipartition. Explicitly, the partially transposed state is not positive-semidefinite if the partially transposed covariance matrix V^{PT} breaks the Heisenberg uncertainty principle [23]:

$$V^{PT} + \frac{i}{2}J < 0. \tag{2.12}$$

which is equivalent to

$$\tilde{\nu}_- < 1/2, \tag{2.13}$$

where $\tilde{\nu}_-$ denotes the smallest symplectic eigenvalue of V^{PT}. Note that the PPT criterion is only sufficient, meaning that even if it is not fulfilled, the state may still be entangled. However, in such a case the entanglement can only be of the so-called *bound*-type, which has little practical relevance [57].

Most importantly for our purposes, the symplectic picture is also equipped with its own, physically justified model of time evolution. In the theory of quantum dynamical semigroups, the state of the system at time $t \geqslant 0$ is governed by the *Gorini–Kossakowski–Lindblad-Sudarshan (GKLS) equation* [60, 61, 62]:

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + D(\hat{\rho}) \tag{2.14}$$

The system Hamiltonian, given by a hermitian non-negative operator $\hat{H}$, is responsible for the isolated-system-like unitary evolution. The *dissipator*, commonly given in its diagonalized form by the non-negative operator

$$D(\hat{\rho}) = \sum_j \left(\hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \frac{1}{2}\{\hat{L}_j^\dagger \hat{L}_j, \hat{\rho}\} \right) \tag{2.15}$$

where $\hat{L}_j$ are the *Lindblad operators*, is responsible for the non-unitary part of the dynamics caused by the coupling of the system to some external environment. The GKLS equation is the most general trace- and complete-positivity-preserving Markovian evolution equation for an open quantum system.

One of the main sources of motivation for studying the symplectic picture is that, due to technical limitations, in practice we are often restricted to Hamiltonians that are polynomials of at most second degree in quadrature operators:

$$\hat{H} = \frac{1}{2}\hat{\vec{\xi}}^T G \hat{\vec{\xi}}, \tag{2.16}$$

where G is a $2N \times 2N$, real, symmetric matrix. Similarly, in practice, the Lindblad operators are typically linear in the quadrature operators [63]:

$$\hat{L}_k = \vec{c}_k \cdot \hat{\vec{\xi}}, \quad \vec{c}_k \in \mathbb{C}^{2N}. \tag{2.17}$$

The resulting evolution preserves the symplectic description: computing the time derivative of the covariance matrix and the vector of means and assuming that the system evolves according to the GKLS equation specified by Eqs. (2.16, 2.17),

the corresponding differential equations for the covariance matrix and the vector of means are [30, 33, 37]

$$
\begin{aligned}
\frac{d}{dt}V &= F_G(V) + F_L(V), \\
\frac{d}{dt}\vec{\xi} &= f_G(\vec{\xi}) + f_L(\vec{\xi}).
\end{aligned}
\tag{2.18}
$$

Here,

$$
\begin{aligned}
F_G(V) &\coloneqq JGV - VGJ, \\
f_G(\vec{\xi}) &\coloneqq JG\vec{\xi},
\end{aligned}
\tag{2.19}
$$

are the Hamiltonian terms, while[2]

$$
\begin{aligned}
F_L(V) &\coloneqq JI_C V + VI_C J + JR_C J^T, \\
f_L(\vec{\xi}) &\coloneqq JI_C \vec{\xi},
\end{aligned}
\tag{2.20}
$$

with $R_C \equiv \mathrm{re}\, C^\dagger C$, $I_C \equiv \mathrm{im}\, C^\dagger C$ and $C_{kl} \coloneqq (\vec{c}_k)_l$, originate from linear Lindblad operators (2.17). We stress that for a generic GKLS equation, the time differentials of V and $\vec{\xi}$ would depend not only on V and $\vec{\xi}$ themselves (and the parameters defining the dissipator), but also third and higher moments, typically leading to an infinite, unsolvable hierarchy of equations. It is only due to the specific choice of the Hamiltonian and Lindblad operators that the equations *close* with respect to the first two moments.

Because the evolution (2.18) preserves the set of degrees of freedom associated with the symplectic picture, and the symplectic picture contains the full information about Gaussian states, it follows that the evolution preserves the set of Gaussian states. For this reason, Eq. (2.18) is often called *Gaussian evolution*. Nonetheless, the symplectic picture and its evolution can be useful also for non-Gaussian input states, accounting for a variety of phenomena, including quantum squeezing and creation of (distillable) entanglement. The only downside is that for non-Gaussian input, the description is incomplete in comparison to the standard quantum formalism. We remark that symplectic time evolution is known to have exact solutions [64, 65, 66] and is well-studied using, e.g. Green functions [67, 68]. Symplectic evolution was also studied by us in the context of stabilizing states against decoherence in two of our papers not included in this dissertation [OP3, OP5].

Due to the ability of the symplectic picture to describe a wide range of phenomena (local and non-local correlations, Heisenberg uncertainty principle, time evolution, etc.) despite being built upon only a limited number of the system's degrees of freedom, it defines a *mesoscopic formalism*, i.e. an intermediate formalism between the full quantum mechanics, and the, in comparison radically simple, classical mechanics. In this thesis, we make heavy use of another mesoscopic formalism, the reduced state of the field. However, before we define it, let us have one more look at symplectic evolution, this time from a different perspective.

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[2]We note the Hamiltonian term and the "linear" dissipative term are typically combined in a single term given by $A = J(G + I_C)$, so that the evolution reads $dV/dt = AV + VA^T + JR_C J^T$. In this work, however, we study the Hamiltonian and the dissipative dynamics separately.

2.2   Bogoliubov transformations

A special class of symplectic evolution is given by *Bogoliubov transformations*, that is, linear transformations $\hat{\vec{\xi}} \to \hat{\vec{\xi'}}$ of the quadrature operators preserving the canonical commutation relations (2.1) [47, 48, 46]. Let us assume a closed system undergoing a unitary transformation:

$$\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^{\dagger}. \tag{2.21}$$

Note that $\hat{U}$ may depend smoothly on time – in such case, it gives rise to *Bogoliubov evolution*. For the transformation/evolution to be of the Bogoliubov-type, $\hat{U}$ must be such that, for a real matrix K,

$$\hat{\xi}'_n = \sum_{m=1}^{2N} K_{nm}\hat{\xi}_m. \tag{2.22}$$

The notation K for the matrix is not a coincidence – in order for the canonical commutation to be satisfied, K must by symplectic, i.e. fulfill Eq. (2.10).
   A slightly different equation arises if instead of the quadrature operators, we consider the annihilation and creation operators

$$\hat{a}_k := \frac{1}{\sqrt{2}}\left(\hat{x}_k + i\hat{p}_k\right), \quad \hat{a}_k^{\dagger} = \frac{1}{\sqrt{2}}\left(\hat{x}_k - i\hat{p}_k\right), \tag{2.23}$$

with the canonical commutation relations (2.1) now reading

$$\left[\hat{a}_k, \hat{a}_{k'}^{\dagger}\right] = \delta_{kk'}, \quad \left[\hat{a}_k, \hat{a}_{k'}\right] = \left[\hat{a}_k^{\dagger}, \hat{a}_{k'}^{\dagger}\right] = 0. \tag{2.24}$$

Since this is the language most often used in the context of Bogoliubov transformations, we adopt it here as well. Collecting the creation and annihilation operators in the vector $\hat{\vec{A}}$ as

$$\hat{A}_n := \begin{cases} \hat{a}_n, & n \in \{1, \ldots, N\} \\ \hat{a}_n^{\dagger}, & n \in \{N+1, \ldots, 2N\} \end{cases}. \tag{2.25}$$

we find that the transformation is of the Bogoliubov-type if and only if

$$\hat{A}'_n := \hat{U}^{\dagger}\hat{A}_n\hat{U} = \sum_{m=1}^{2N} \mathcal{X}_{nm}\hat{A}_m, \tag{2.26}$$

with $\mathcal{X}$ a complex matrix fulfilling the *complex symplectic property* [69, 70]:

$$\mathcal{X}\mathcal{S}\mathcal{X}^{\dagger} = \mathcal{S}, \tag{2.27}$$

where $\mathcal{S} = \operatorname{diag}\left[\mathbb{1}_N, -\mathbb{1}_N\right]$ and $\mathbb{1}_N$ denotes the $N \times N$ identity matrix. Due to the symplectic property,

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_{\uparrow} & \mathcal{X}_{\downarrow} \\ \mathcal{X}_{\downarrow}^* & \mathcal{X}_{\uparrow}^* \end{bmatrix}, \tag{2.28}$$

where $\mathcal{X}_\updownarrow$ are arbitrary complex matrices of size $N \times N$. Note that the symplectic condition (2.27) is nothing more but a complexification of Eq. (2.10), a consequence of replacing the real-valued quadrature operators by the creation and annihilation operators. The two conditions are formally equivalent.

In the more general case of an open system, the total density operator of the system and environment is as well transformed according to Eq. (2.21). However, we are only interested in the state of the system, given by a partial trace over the degrees of freedom of the environment:

$$\hat{\rho}_S = \mathrm{Tr}_E\, \hat{\rho}. \tag{2.29}$$

The Bogoliubov transformation itself (2.26) remains the same. Still, assuming the system and the environment span N_S and N_E modes respectively, it is convenient to additionally split the matrices entering the block decomposition (2.28) into

$$\mathcal{X}_\uparrow = \begin{bmatrix} \mathcal{X}_{\uparrow S} & \mathcal{X}_{\uparrow C} \\ \mathcal{X}_{\uparrow C'} & \mathcal{X}_{\uparrow E} \end{bmatrix}, \quad \mathcal{X}_\downarrow = \begin{bmatrix} \mathcal{X}_{\downarrow S} & \mathcal{X}_{\downarrow C} \\ \mathcal{X}_{\downarrow C'} & \mathcal{X}_{\downarrow E} \end{bmatrix}, \tag{2.30}$$

where $\mathcal{X}_{\updownarrow S}$ is an $N_S \times N_S$ matrix associated with the system, $\mathcal{X}_{\updownarrow E}$ is an $N_E \times N_E$ matrix associated with the environment, and $\mathcal{X}_{\updownarrow C}$, $\mathcal{X}_{\updownarrow C'}$ are appropriately-sized matrices associated with both. Note that the case of the closed system can be retrieved easily by setting $N_E = 0$ (which, in particular, implies $\mathcal{X}_\updownarrow = \mathcal{X}_{\updownarrow S}$) and dropping the then-redundant lower indices S.

2.2.1 Dynamical Casimir effect

From a formal point of view, Bogoliubov transformations essentially define a change of basis of the Hilbert space. For this reason, their applicability is near universal, including solid-state physics, quantum optics and quantum field theory [47, 48, 49, 50, 51]. Bogoliubov transformations were also used by us in the context of optical metrology in [OP1, OP2]. One of the most celebrated phenomenons originating from a Bogoliubov transformation is the *dynamical Casimir effect*, a generalization of the Casimir effect [71, 72, 73, 74] defined by a spontaneous production of particles in a medium following from non-trivial time dependence of either its boundary or its material coefficients [52, 53, 54, 55].

A particularly interesting case of the dynamical Casimir effect takes place in an accelerating medium, a case unique due to the absence of medium boundaries [75, 76]. Consider the Maxwell equations in vacuum [21, 22]:

$$\partial_t \vec{D}(\vec{r}, t) = \vec{\nabla} \times \vec{H}(\vec{r}, t), \quad \vec{\nabla} \cdot \vec{D}(\vec{r}, t) = 0,$$
$$-\partial_t \vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{E}(\vec{r}, t), \quad \vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0. \tag{2.31}$$

where $\vec{D}$ and $\vec{E}$ describe the electric field and $\vec{B}$ and $\vec{H}$ describe the magnetic field. Solving these equations in the Heisenberg picture for the case of a medium, which is accelerating for some set period $t \in [0, T]$, eventually leads to a Bogoliubov transformation of the form [75]

$$\hat{a}_{R,\mathrm{out}}(\vec{k}) = e^{-i\phi} \left[f_{R+} \hat{a}_{R,\mathrm{in}}(\vec{k}) + f_{R-} \hat{a}_{L,\mathrm{in}}^\dagger(-\vec{k}) \right],$$
$$\hat{a}_{L,\mathrm{out}}^\dagger(-\vec{k}) = e^{-i\phi} \left[f_{L+} \hat{a}_{R,\mathrm{in}}(\vec{k}) + f_{L-} \hat{a}_{L,\mathrm{in}}^\dagger(-\vec{k}) \right], \tag{2.32}$$

where $(\hat{a}_{R,\text{in}}, \hat{a}_{R,\text{out}})(\vec{k})$ and $(\hat{a}_{L,\text{in}}, \hat{a}_{L,\text{out}})(-\vec{k})$ are the initial and final annihilation operators for right- and left-helicity photons with wave vectors $\pm\vec{k}$, while ϕ, $f_{L\pm}$, $f_{R\pm}$ are certain functions of time originating from solving the Maxwell equations.[3]

By considering system initially in the vacuum and computing the photon number densities after the motion, we obtain

$$
\begin{aligned}
\langle \hat{n}_R(T) \rangle &= \langle 0|\hat{a}_{R,\text{out}}^\dagger(\vec{k})\hat{a}_{R,\text{out}}(\vec{k})|0\rangle = |f_{R-}|^2\delta(0), \\
\langle \hat{n}_L(T) \rangle &= \langle 0|\hat{a}_{L,\text{out}}^\dagger(\vec{k})\hat{a}_{L,\text{out}}(\vec{k})|0\rangle = |f_{L+}|^2\delta(0),
\end{aligned}
\tag{2.33}
$$

where $\delta(0)$ is the Dirac delta singularity. As was verified in [75, 76], at least for some wave vectors these numbers are monotonically growing functions of T. Therefore, the motion of the medium results in a potentially unbounded particle production in the vacuum and, hence, the prediction of the dynamical Casimir effect.

2.3 Reduced state of the field

A different mesoscopic formalism was recently introduced in [45] in the form of the *reduced state of the field (RSF)*. The main idea was to describe macroscopic bosonic fields, such as electromagnetic fields or gravitational waves, which are usually considered to be semiclassical, from the quantum point of view.

Let us go back to the GKLS equation (2.14) and consider the dissipator in its most general, non-diagonal form:

$$
\frac{d}{dt}\hat{\rho} = -i\left[\hat{H}, \hat{\rho}\right] + \sum_{k,k'} B_{kk'} \left(\hat{J}_k \hat{\rho} \hat{J}_{k'}^\dagger - \frac{1}{2}\left\{ \hat{J}_{k'}^\dagger \hat{J}_k, \hat{\rho} \right\} \right).
\tag{2.34}
$$

Here, $\hat{J}_k$ are the jump operators and B is a non-negative matrix. [The original equation (2.14) is obtained by diagonalizing the non-Hamiltonian part.] In the formalism of RSF, it is assumed that macroscopic fields can be approximately treated as a set of individual particles subject to spontaneous decay and production, as well as interaction with coherent classical sources and random scattering by the environment.

In such a setting, the arising Hamiltonian is [45]:

$$
\hat{H} = \sum_{k=1}^{N} \left(\omega_k \hat{a}_k^\dagger \hat{a}_k + i\zeta_k \hat{a}_k^\dagger - i\zeta_k^* \hat{a}_k \right).
\tag{2.35}
$$

Here, the first term is nothing but the base Hamiltonian for bosonic fields, with the positive frequencies ω_k defining the energy levels of the system. The remaining two terms correspond to an interaction of the input state with an N-mode coherent state $|\vec{\zeta}\rangle$ (which can be considered classical for large amplitudus) in an asymmetric beam-splitter [77].

For the dissipator, three families of jump operators were considered [45]:

- spontaneous decay of particles in the field, given by

$$
\hat{J}_k = \hat{a}_k, \quad B_{kk'} = \Gamma_\downarrow^{k'k}
\tag{2.36}
$$

---

[3]For the (lengthy) details we refer the Reader to our Third Paper.

- spontaneous creation of particles in the field, given by

$$\hat{J}_k = \hat{a}_k^\dagger, \quad B_{kk'} = \Gamma_\uparrow^{k'k} \tag{2.37}$$

- random scattering [78, 79, 80], given by

$$\hat{J}_k = \hat{U}_k, \quad B_{kk'} = \eta_k \delta_{kk'} \tag{2.38}$$

with $\eta_k \geqslant 0$, $\sum_k \eta_k = 1$ and $\hat{U}_k$ unitary operations producing simplified Bogoliubov transformations (2.26) of the form

$$\hat{U}_k^\dagger \hat{a}_m \hat{U}_k = \sum_{l=1}^{N} (u_k)_{ml} \hat{a}_l, \tag{2.39}$$

where u_k are unitary matrices themselves (of size $N \times N$).

Similarly to how the dynamics assumed in the previous section preserve the symplectic description $(V, \vec{\xi})$, the dynamics assumed here for the macroscopic fields preserve RSF $(r, |\alpha\rangle)$, consisting of two objects: the *single-particle density matrix*

$$r := \sum_{k,k'=1}^{N} \text{Tr}\left[\hat{\rho}\,\hat{a}_{k'}^\dagger \hat{a}_k\right] |k\rangle\langle k'| \tag{2.40}$$

and the *averaged field*

$$|\alpha\rangle := \sum_{k=1}^{N} \text{Tr}\left[\hat{\rho}\,\hat{a}_k\right] |k\rangle. \tag{2.41}$$

Just like the symplectic picture, RSF forms a self-contained, mesoscopic formalism. Like the covariance matrix V, the single-particle density matrix r contains the simplest non-local information about the system, with the averaged field $|\alpha\rangle$ containing local information akin to vector of means $\vec{\xi}$. Note that the RSF components are of dimension N only.

Other properties of quantum systems also translate naturally to RSF. For example, RSF is equipped with a dedicated measure of entropy. Derived from the von Neumann entropy (we set $k_B = 1$)

$$S_V(\hat{\rho}) := -\text{Tr}\,\hat{\rho}\ln\hat{\rho} \tag{2.42}$$

through the *maximum entropy principle* [81, 82], the *reduced (von Neumann) entropy* equals [45]

$$s_v(\hat{\rho}) = \text{tr}[(r_\alpha + \mathbb{1}_N)\ln(r_\alpha + \mathbb{1}_N) - r_\alpha \ln r_\alpha], \tag{2.43}$$

where r_α defines the non-negative *correlation matrix*:

$$r_\alpha := r - |\alpha\rangle\langle\alpha| \geqslant 0. \tag{2.44}$$

The reduced entropy satisfies the natural condition $s_v(\hat{\rho}) \geqslant 0$, with equality if and only if the correlation matrix is equal to zero, which happens only when the

density operator of the field is given by a coherent state. In contrast, the von Neumann entropy vanishes for any pure state.

Importantly, due to the fact that RSF is preserved by the dynamics given by (2.35) and the three families of jump operators (2.36-2.37), substituting these into the GKLS equation (2.34) leads to closed evolution equations for RSF, dubbed the *reduced kinetic equations* [45]

$$
\begin{aligned}
\frac{d}{dt}r = &- i\big[h, r\big] + |\zeta\rangle\langle\alpha| + |\alpha\rangle\langle\zeta| \\
&+ \frac{1}{2}\big\{\gamma_\uparrow - \gamma_\downarrow, r\big\} + \gamma_\uparrow \\
&+ \sum_k \eta_k\big(u_k r u_k^\dagger - r\big), \\
\frac{d}{dt}|\alpha\rangle = &- ih|\alpha\rangle + \frac{1}{2}\big(\gamma_\uparrow - \gamma_\downarrow\big)|\alpha\rangle + |\zeta\rangle \\
&+ \sum_k \eta_k\big(u_k - \mathbb{1}_N\big)|\alpha\rangle.
\end{aligned}
\tag{2.45}
$$

Here,

$$
h := \sum_{k=1}^N \omega_k |k\rangle\langle k|,
\tag{2.46}
$$

$$
|\zeta\rangle := \sum_{k=1}^N \zeta_k |k\rangle,
\tag{2.47}
$$

$$
\gamma_\updownarrow := \sum_{k,k'=1}^N \Gamma_\updownarrow^{kk'} |k\rangle\langle k'|
\tag{2.48}
$$

are the single-particle counterparts to $\hat{H}$, $\vec{\zeta}$ and $\Gamma_\updownarrow$, respectively, while u_k are fixed by Eq. (2.39). Let us stress that, in an analogy to symplectic evolution (2.18), the reduced kinetic equations are closed with respect to RSF: the time differentials of r and $|\alpha\rangle$ depend only on r and $|\alpha\rangle$ themselves and not other correlation functions. This is true only because of the specific choice of dynamics assumed for the system.

Looking at the definitions of the single particle density matrix (2.40) and the averaged field (2.41), and remembering that the creation and annihilation operators are related to the quadrature operators by a linear transformation (2.23), it is clear that RSF is built solely upon first and second moments, and therefore has to be strictly contained within the symplectic picture (which encompasses all such moments). Counting the (real) degrees of freedom for an N-mode RSF, we get only $N^2 + 2N$, compared to the previously obtained $2N^2 + 3N$ for the symplectic picture. A natural question arises: what information is missing in RSF compared to the symplectic description? Specifically, what physical processes described by symplectic evolution (2.18) cannot be described by the reduced kinetic equations (2.35)? Furthermore, how to interpret the reduced von Neumann entropy (2.43)? Is the Heisenberg uncertainty principle (2.8) contained within RSF? As we will see, answering these questions will be a key step in our identification of symplectic evolution's classicality.

Chapter 3

Summary of dissertation

3.1 *Dissipative evolution of quantum Gaussian states* [First Paper]

As said above, due to its full reliance on first and second moments only, the RSF formalism has to be contained within the symplectic picture. However, a close inspection of the two frameworks' evolution equations leads to a discrepancy with this view. The dissipator in the symplectic evolution equation (2.18) is derived from the Lindblad operators (2.17), which are linear in the quadrature operators. And although the RSF's first two families of jump operators (2.36-2.37) are also linear, and therefore necessarily compatible with the symplectic picture, the last family consists of jump operators which are unitary (2.38), and are thus beyond the assumed model of symplectic evolution. This means that symplectic evolution, as given by Eq. (2.18), is incomplete, in the sense that there must exist unitary Lindblad operators compatible with it. Before we can analyze the classicality of symplectic evolution, we therefore have to first find and give meaning to this missing component. This is exactly the goal of our First Paper, whose main results are summarized by us in this section.

3.1.1 Dissipative evolution stemming from unitary Lindblad operators

Let us go back to the GKLS equation (2.14). Being interested in the dissipative part of the equation only, we disregard the Hamiltonian term. As for the dissipator, we follow the seminal papers within the field of quantum dynamical semigroups [78, 79, 80], as well as some more recent works [45, 83], and consider the particular case of M Lindblad operators, all proportional to unitary operators:

$$\hat{L}_j = \sqrt{\gamma_j}\hat{U}_j, \tag{3.1}$$

where we introduce $\gamma_j \geqslant 0$ to account for different dissipation strengths from different Lindblad operators. The corresponding GKLS equation reads as

$$\frac{d}{dt}\hat{\rho} = \sum_{j=1}^{M} \gamma_j \left(\hat{U}_j\hat{\rho}\hat{U}_j^\dagger - \hat{\rho} \right). \tag{3.2}$$

For convenience, we assume that γ_j are normalized as $\sum_{j=1}^{M} \gamma_j = 1$.

As we show in our First Paper, if

$$\hat{U}_j = e^{-i\hat{h}_j}, \tag{3.3}$$

for some hermitian operators $\hat{h}_j$ being polynomials of at most second degree in quadrature operators, the resulting evolution (3.2) is indeed compatible with the symplectic picture. As is well known, every unitary operator of the form produces a Bogoliubov transformations according to Eq. (2.22). Based on this, our First Paper proves that Eq. (3.2) implies

$$\frac{d}{dt}V = \sum_{j=1}^{M} \gamma_j \left[K_j V K_j^T - V + F_j(\vec{\xi}) \right],$$

$$\frac{d}{dt}\vec{\xi} = \sum_{j=1}^{M} \gamma_j K_j \vec{\xi}, \tag{3.4}$$

where

$$F_j(\vec{\xi}) = (K_j - \mathbb{1}_{2N})\vec{\xi}\vec{\xi}^T(K_j^T - \mathbb{1}_{2N}). \tag{3.5}$$

As seen, the equations are closed with respect to $(V, \vec{\xi})$.

To simplify our considerations, we note that in typical applications of symplectic evolution, concerning e.g. quantum entanglement, the vector of mean values is irrelevant. For this reason, later on we assume $\vec{\xi}(0) = 0$, in which case $F_j(\vec{\xi}) = 0$[1] and the evolution simplifies to

$$\frac{d}{dt}V = \sum_{j=1}^{M} \gamma_j \left[K_j V K_j^T - V \right]. \tag{3.6}$$

The striking similarity between the above equation and the last term in Eq. (2.45) tells us that the former is exactly the component of symplectic evolution that was missing in comparison to the reduced kinetic equations. It remains to see what this new component means from the physical point of view, so that we can successfully assess its classicality later. Following our First Paper, below we will demonstrate that depending on the number and nature of the unitary Lindblad operators, Eq. (3.2) and its symplectic representation (3.6) can have radically different applications, ranging from random scattering to engineered dissipation.

Before we do that, however, let us briefly discuss the issue of Gaussianity of the above evolution. As said previously, one of the main motivations for the study of Gaussian states is that this is exactly the group of states preserved by symplectic evolution, as given by Eq. (2.18). However, as observed recently within the context of resource theories of non-Gaussianity [25, 26], the same experimental tools that allow for working with Gaussian states are viable for working with *quantum Gaussian states*, a generalization of the set of Gaussian states consisting

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[1]To see this, consider the bottom line of Eq. (3.6). Solving for the vector of means, we immediately obtain $\vec{\xi}(t) = \exp[\Sigma_{j=1}^{M} \gamma_j K_j t]\vec{\xi}(0)$, which vanishes for $\vec{\xi}(0) = 0$. Obviously, this implies $F_j(\vec{\xi}) = 0$.

of Gaussian states and their convex combinations [84, 85, 86]. As we show in our
First Paper, this is exactly the set of states preserved by the dissipative evolution
stemming from the unitary operators (3.3). Therefore, such evolution is just as
natural to consider in experiment as the standard symplectic dynamics. In fact,
both forms of dynamics can be freely combined, promoting the usual Gaussian
evolution to a quantum Gaussian one. For example, one may introduce generic
quantum Gaussian noise into an otherwise Gaussian system, as considered by us
in the context of counteracting decoherence in one of our other papers [OP3].

3.1.2   Random scattering

The original articles [79, 80] that inspired our choice of unitary Lindblad operators
associate such choice with random scattering. In our First Paper, we argue further
behind this interpretation by employing the *collision model* [87, 88, 89]. In this
model, the initial system is coupled to an infinite number of identical copies of
ancilla $\hat{\eta}$. The total initial state is separable:

$$\hat{\rho}_T(0) = \hat{\rho}(0) \otimes \hat{\eta} \otimes \hat{\eta} \otimes \dots \tag{3.7}$$

During each time step Δt, a unitary operation $\hat{W}_1$ acts on the system and the first
ancilla, after which the latter is traced out. Since the corresponding total state
has the same form as initially (3.7):

$$\hat{\rho}_T(\Delta t) = \hat{\rho}(\Delta t) \otimes \hat{\eta} \otimes \hat{\eta} \otimes \dots \tag{3.8}$$

the second and further steps are fully analogous. This results in a recurrence
relation

$$\hat{\rho}(n\Delta t) = \mathrm{Tr}_\eta\{\hat{W}_n[\hat{\rho}[(n-1)\Delta t] \otimes \hat{\eta}]\hat{W}_n^\dagger\}. \tag{3.9}$$

For our purposes, we adopt the most general general version of the collision model
[88], in which the unitary operators $\hat{W}_n$ are unrestricted:

$$\hat{W}_n = \mathcal{T}\exp\left(-i\int_{(n-1)\Delta t}^{n\Delta t} d\tau\, \hat{w}_n(\tau)\right) \tag{3.10}$$

where $\mathcal{T}$ is the time-ordering operator and the time-dependent Hamiltonian $\hat{w}_n$ can
act on both the system and the n-th ancilla in an arbitrary way. Through a proper
choice of the ancilla and the unitaries, the collision model can emulate a wide range
of dynamics, making it a popular tool in studies of optics, thermodynamics, non-
Markovianity, and many others [88, 89].

As we show in our First Paper, the collision model reproduces the evolution
equation (3.2) if the ancillas are chosen to be qu*d*its of dimension $d = M + 1$ in
the ground state,

$$\hat{\eta} = |0\rangle\langle 0|, \tag{3.11}$$

and $\hat{W}_n$ are generated by the following Hamiltonian:

$$\hat{w}_n(\tau) = \hat{o}_n(\tau) + \sum_{m=-\infty}^{\infty}\sum_{j=1}^{M}\delta(\tau - m\Delta T)\hat{h}_j \otimes |j\rangle\langle j|. \tag{3.12}$$

Here, $\hat{o}_n$ is defined to be a generator of an orthogonal transformation

$$\hat{O}(\Delta t) = \mathcal{T} \exp\left(-i \int_{(n-1)\Delta t}^{n\Delta t} d\tau \, \hat{o}_n(\tau)\right) \tag{3.13}$$

such that its action on the ancilla is

$$\hat{O}(\Delta t)|0\rangle = \sqrt{1 - \Delta t}|0\rangle + \sqrt{\Delta t} \sum_{j=1}^{M} \sqrt{\gamma_j}|j\rangle. \tag{3.14}$$

Furthermore, $\delta(\tau - m\Delta T)$ is the Dirac delta distribution centered at the point $\tau = m\Delta T$, while $\hat{h}_j$ are the generators of the unitary transformations entering the evolution equation (3.2), i.e. they are given by Eq. (3.3).

Notably, the Hamiltonian (3.12) has a well-known structure of the *kicked top* [90, 91, 92], a model generally characterized by Hamiltonians consisting of standard, smooth hermitian generators (in our case given by $\hat{o}_n$) periodically disturbed (in our case with period length ΔT) by a Dirac-delta potential, leading to chaotic behaviour. Due to its relative simplicity and ease of implementation in terms of qubits, the kicked top is the theoretical [92] and experimental [93] go-to model for testing the implications of dynamical chaos on quantum phenomena (such as, e.g. entanglement).

Let us now examine how these results connect to random scattering. Each collision can be seen as a single scattering event in the medium described by ancillas in the state (3.14). Crucially, the probability that the system will be kicked by the j-th Hamiltonian $\hat{h}_j$ depends on γ_j through eqs. (3.12, 3.14). For a large number of mutually non-commuting unitaries, the uncertainty in the outcome state is large. In particular, in the limit of infinite unitaries the outcome probabilities γ_j may be replaced by a probability measure $\mu(dU)$ on the unitary group, yielding a scattering integral [78]

$$\frac{d}{dt}\hat{\rho} = \int d\mu(U) \left(\hat{U}\hat{\rho}\hat{U}^\dagger - \hat{\rho}\right). \tag{3.15}$$

Meanwhile, because the ancillas are traced out after each collision, the state of the bath is constant throughout the whole interaction, fulfilling the expectation that scattering should not affect the bath. These results are consistent with previous findings [80, 94] that unitary Lindblad operators can be interpreted as the S-matrices of a system interacting with a dilute gas.

As a final remark, we observe that, as shown explicitly in our First paper, for an initial Gaussian state the time-evolved state in the discussed equation is quantum Gaussian. Combining this with the above analysis, we can see that the latter family can be obtained from the former by simply subjecting it to random scattering, which may be regarded as pure noise. This further strengthens the practical connection between Gaussian and quantum Gaussian states, in particular supporting the developments made over the last decade to construct measures of quantum non-Gaussianity [95, 96, 97], which, contrary to the more traditional measures of non-Gaussianity [98, 99, 100, 101, 102] do not assign positive values of the resource to convex combination of Gaussian states.

3.1.3 Entanglement creation in two-mode states

As a final major result of our First Paper, we consider an engineered dissipation scenario, in which we use the discussed evolution equation for creation of two-mode entanglement from a system initially in the vacuum state

$$\hat{\rho}(0) = |00\rangle\langle 00|, \tag{3.16}$$

which is separable and Gaussian. For the evolution, we choose a single Lindblad operator from the one-parameter family of unitary two-mode squeezing operators

$$\hat{L} = \hat{U}_r := e^{i\hat{h}_r}, \quad \hat{h}_r = -ir(\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2), \tag{3.17}$$

where $r > 0$ is the squeezing strength. Let us stress that, from the physical point of view, the evolution given by such a Lindblad operator is not at all equivalent to a "smooth" unitary evolution given by a squeezing Hamiltonian $\hat{H} = \hat{h}_r$. Instead, here, the squeezing should be understood as a series of regular, infinitely strong but infinitesimally short squeezing kicks, as discussed in the previous section.

To fulfill our goal, our First Paper first derives explicit solutions to both Eqs. (3.2, 3.6) for the general case of a single Lindblad operator. Then, these results are applied to the specific case at hand, yielding the following time-evolved covariance matrix of the system:

$$V(t) = \begin{bmatrix} A(t) & C(t) \\ C(t) & A(t) \end{bmatrix}, \tag{3.18}$$

where

$$\begin{aligned}
A(t) &= \frac{1}{2}e^{2t\sinh^2 r}\cosh\left(t\sinh 2r\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
C(t) &= \frac{1}{2}e^{2t\sinh^2 r}\sinh\left(t\sinh 2r\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\end{aligned} \tag{3.19}$$

We then proceed in two steps. In the first step, we use the PPT criterion (2.12) to certify that the state is entangled. To this end, we first calculate the smallest symplectic eigenvalue ν of the partially transposed covariance matrix, finding

$$\tilde{\nu}_-(t) = \frac{1}{2}\exp\left[-\left(1 - e^{-2r}\right)t\right]. \tag{3.20}$$

It is easy to see that this always satisfies (2.13), meaning that the state is entangled at all times $t > 0$.

In the second step, we assess how much entanglement is contained in the time-evolved state. Note that even though the initial state is Gaussian, the time-evolved state is only quantum Gaussian (i.e. it is a convex combination of Gaussian states), meaning that we cannot use one of the (relatively easy to calculate) Gaussian measures of entanglement. Instead, we consider *squashed entanglement*, which for a generic bipartite state $\hat{\sigma}_{AB}$ is defined as [57, 103, 104]

$$\mathcal{E}_{\text{sq}}(\hat{\sigma}_{AB}) := \frac{1}{2}\inf_{\hat{\sigma}_{ABE}} I(A : B|E), \tag{3.21}$$

where $I(A : B|E) := S_V(\hat{\sigma}_{AE}) + S_V(\hat{\sigma}_{BE}) - S_V(\hat{\sigma}_E) - S_V(\hat{\sigma}_{ABE})$ is the conditional mutual information, $\hat{\sigma}_X$ are the (reduced) density operators of (sub)systems X and the minimization is over all pure states $\hat{\sigma}_{ABE}$ such that $\hat{\sigma}_{AB} = \text{Tr}_E \hat{\sigma}_{ABE}$. Like other entanglement measures defined in terms of optimization over a set of states, squashed entanglement is notoriously difficult to calculate [105]. Therefore, we instead compute a lower bound for it and show that it is an asymptotically unbounded function of time (corresponding to maximal possible amount of entanglement).

We begin by observing that, due to the extremality of Gaussian states with respect to continuous, superadditive entanglement measures [106], the squashed entanglement of any state $\hat{\sigma}$ is lower-bounded by the squashed entanglement of a Gaussian state $\hat{\sigma}_G$ with the same covariance matrix. Furthermore, squashed entanglement of any state is lower-bounded by so-called distillable entanglement $\mathcal{E}_{\text{dist}}$ [103], which, in turn, is lower-bounded by the coherent information [107, 108]

$$I_{\mathcal{C}}(\hat{\sigma}) := S_V(\hat{\sigma}_A) - S_V(\hat{\sigma}), \tag{3.22}$$

where $\hat{\sigma}_A = \text{Tr}_B \hat{\sigma}$. This means that we have the following chain of inequalities

$$\mathcal{E}_{\text{sq}}[\hat{\rho}(t)] \geqslant \mathcal{E}_{\text{sq}}[\hat{\rho}_G(t)] \geqslant \mathcal{E}_{\text{dist}}[\hat{\rho}_G(t)] \geqslant I_{\mathcal{C}}[\hat{\rho}_G(t)] = S_V[\hat{\rho}_{G,A}(t)] - S_V[\hat{\rho}_G(t)], \tag{3.23}$$

where $\hat{\rho}_G(t)$ is a Gaussian state with the same covariance matrix (3.18) as our state and $\hat{\rho}_{G,A}(t) = \text{Tr}_B \hat{\rho}_G(t)$. Crucially, both von Neumann entropies on the r.h.s. are relatively simple functions of the symplectic eigenvalues of the respective state [109]. Calculating them explicitly for asymptotically large times, our First Paper finally leads to the following bound:

$$\mathcal{E}_{\text{sq}}[\hat{\rho}(t)] \geqslant -2 + 4t \sinh^2(r)[\coth(r) - 1], \tag{3.24}$$

Clearly, the r.h.s. is a linear function in t with positive slope, since $\coth(r) > 1$ for all $r > 0$. Therefore, the r.h.s. is asymptotically infinite, and thus the same is also true for squashed entanglement itself. This is what we wanted to show.

3.2 *Reduced state of the field and classicality of quantum Gaussian evolution* [Second Paper]

Having found the missing component of symplectic evolution, we are ready to compare the associated formalism with RSF. This will eventually lead us to an interpretation of RSF as a semiclassical framework, and in turn let us identify precisely the semiclassical subset of symplectic evolution. The obtained results form the basis of our Second Paper, which we briefly summarize in this section.

3.2.1 Reduced state of the field as a classical description of bosonic fields

To see what physical information is associated with the degrees of freedom contained within RSF, our Second Paper first shows that RSF is related to the symplectic picture via

$$r = \mathcal{R}(V + \vec{\xi}\vec{\xi}^\dagger)\mathcal{R}^\dagger - \frac{1}{2}\mathbb{1}_N, \quad |\alpha\rangle = \mathcal{R}\vec{\xi}, \tag{3.25}$$

where we define

$$\mathcal{R} \coloneqq \frac{1}{\sqrt{2}}\sum_{k=1}^{N}|k\rangle\big[\langle 2k-1| + i\langle 2k|\big] \tag{3.26}$$

as the *reduction matrix*.[2] Furthermore, we show that the Heisenberg uncertainty principle (2.9) translates to non-negativity of the correlation matrix (2.44). Note that it follows immediately from Eq. (3.25) that

$$r_\alpha = \mathcal{R}V\mathcal{R}^\dagger - \frac{1}{2}\mathbb{1}_N. \tag{3.27}$$

As we demonstrated previously by simply counting the associated degrees of freedom, RSF must contain restricted information with respect to the symplectic picture. The relation (3.27), together with the PPT criterion (2.12), let us find out exactly what the nature of this missing information is:

Proposition 1. *The RSF framework contains no information about bipartite distillable entanglement.*

The proof of the above statement relies on one key observation: as we show in our Second Paper, among all the covariance matrices corresponding to a fixed correlation matrix through Eq. (3.27), there is always one whose all partial transpositions fulfill the Heisenberg uncertainty principle (2.12). This means that every distillably entangled state's RSF is identical to the RSF of some fully separable

---

[2]Note that for the purposes of our Second Paper, it was more convenient to denote the covariance matrix by V_{cov}, with V defined as $V \coloneqq V_{\text{cov}} - \vec{\xi}\vec{\xi}^\dagger$, i.e. shifted in comparison to Eq. (2.5), so that it was not dependent on the first moments. Here, for clarity, we follow the standard convention (2.5) throughout the whole thesis, which however means that a few formulas presented in this section differ slightly from their analogs in our Second Paper.

state. This makes the PPT criterion, and therefore the question of whether the state contains bipartite distillable entanglement in the system, undecidable based on the information contained in the RSF framework alone.

Proposition 1 takes an even stronger form for two-mode Gaussian states, for which the PPT criterion is equivalent to the presence of any form of entanglement, not only distillable entanglement [57]:

Corollary 2. *In the case of two-mode Gaussian states, the RSF framework contains no information about any form of entanglement.*

Based on these results, our Second Paper conjectures that, more generally, RSF contains no information about any type of quantum entanglement in any quantum state. Irrespectively, Proposition 1 and Corollary 2 show that the ability to describe entanglement within the RSF formalism is severely limited, strongly suggesting the framework to be semiclassical.

The semiclassicality of RSF becomes even more apparent after taking a closer look at the reduced entropy (2.43). To see this, let us compare it with a "competitor" of the von Neumann entropy (2.42), the Wehrl entropy

$$S_W(\hat{\rho}) := - \int \frac{d^{2N}\vec{\beta}}{\pi^N} Q(\vec{\beta}) \ln Q(\vec{\beta}). \tag{3.28}$$

Here, $Q(\vec{\beta}) = \langle \vec{\beta} | \hat{\rho} | \vec{\beta} \rangle$ is the Husimi Q representation [110] of the state, $|\vec{\beta}\rangle$ is an N-mode coherent state and the integration is over the whole complex plane $\mathbb{C}^N$. The Wehrl entropy is typically considered to be a semiclassical approximation to the von Neumann entropy, since it is constructed from the latter by replacing the density operator by its representation $Q(\vec{\beta})$ in the phase-space [111, 112]. This results in significant differences: unlike the von Neumann entropy, the Wehrl entropy attains its minimum value, N, only for coherent states [113], and it is invariant only under some unitary transformations of the state.

Looking at the reduced entropy (2.43), we can see that it possesses the same qualities. The fact that it is minimized by coherent states was already discussed. Furthermore, as we check in our Second Paper, it is also not invariant under all unitary operations. Finally, we note that by construction, the reduced entropy provides an upper bound to the von Neumann entropy, another quality shared with the Wehrl entropy. To make this point even stronger, in our Second Paper, we use the same maximum entropy principle that was originally employed [45] in derivation of the reduced entropy to produce a new entropy of RSF based on the Wehrl entropy, which we call the *reduced Wehrl entropy*:

$$s_w(\hat{\rho}) := \operatorname{tr} \ln(r_\alpha + \mathbb{1}_N) + N. \tag{3.29}$$

We observe that the reduced Wehrl entropy has similar qualitative properties to the original reduced (von Neumann) entropy, e.g. it is invariant under the same unitary transformations and is minimized by coherent states. Importantly, as proved in our Second Paper, the two reduced entropies can also be linked quantitatively:

Proposition 3. *The following relation between the RSF entropies holds:*

$$0 < s_w(\hat{\rho}) - s_v(\hat{\rho}) \leqslant N. \tag{3.30}$$

For states with mean particle number much bigger than the effective number of modes $\operatorname{tr} r = \langle \hat{n} \rangle \gg N$, the term N is vanishing in comparison to s_w, s_v. Therefore, it follows from Eq. (3.30) that for most many-particle states, the two reduced entropies are approximately equal. In conclusion, the RSF entropies based on the "quantum" von Neumann and on the "classical" Wehrl entropy are nearly identical to each other and akin to the Wehrl entropy.[3] This cements the classicality of the RSF formalism.

3.2.2   Classicality of quantum Gaussian evolution

Having established the classicality of RSF, we can conclude that, since they preserve the formalism's semiclassical degrees of freedom, the reduced kinetic equations (2.45) are semiclassical themselves. In turn, this means that symplectic evolution can be considered classical if, when subjected to the reduction map (3.25), it takes the form of the reduced reduced kinetic equations. Based on this principle, our Second Paper derives the exact conditions for classicality of symplectic evolution, specifically quantum Gaussian evolution, as defined by Eqs. (2.18, 3.6). For clarity, we consider each of the three terms entering the quantum Gaussian evolution equations separately.

Proposition 4. *Let $(V, \vec{\xi})$ denote the symplectic description of a system undergoing the quantum Gaussian evolution*

$$
\frac{d}{dt}V = \begin{cases} F_G(V), \\ F_L(V), \\ F_U(V), \end{cases} \qquad \frac{d}{dt}\vec{\xi} = \begin{cases} f_G(\vec{\xi}), \\ f_L(\vec{\xi}), \\ f_U(\vec{\xi}), \end{cases} \tag{3.31}
$$

where (F_G, f_G) [defined in Eq. (2.19)] correspond to Hamiltonian symplectic evolution, (F_L, f_L) [defined in Eq. (2.20)] correspond to dissipative symplectic evolution originating from linear Lindblad operators (2.17), and, finally,

$$
\begin{aligned}
F_U(V) &:= \sum_j \gamma_j \left(K_j V K_j^T - V \right), \\
f_U(\vec{\xi}) &:= \sum_j \gamma_j \left(K_j \vec{\xi} - \vec{\xi} \right),
\end{aligned} \tag{3.32}
$$

correspond to dissipative symplectic evolution (3.4) originating from unitary Lindblad operators (2.39).

The evolution can be written as reduced kinetic equations (2.45) and is thus classical with respect to the RSF formalism if and only if[4]

$$
\begin{cases} 0 = \left[J, G \right] & \text{for } (F_G, f_G), \\ 0 = \left[J, I_C \right] & \text{for } (F_L, f_L), \\ 0 = \left[J, K_j \right] & \text{for } (F_U, f_U). \end{cases} \tag{3.33}
$$

---

[3]Note that a similar result does not hold for the original entropies, as while the difference $S_W - S_V$ is always positive, there is no known upper bound for it.

[4]In our Second Paper, the last condition was given as "$0 = \mathcal{R} K_j \mathcal{R}^T$ and $\mathcal{R} K_j \mathcal{R}^\dagger$ is unitary". However, we have since realized that this simplifies to "$0 = [J, K]$", which can be proven in the same way as the other conditions.

The corresponding reduced kinetic equations are governed by

$$
\begin{cases}
h = i\mathcal{R}JG\mathcal{R}^\dagger & \text{for } (F_G, f_G), \\
\gamma_\updownarrow = \pm\mathcal{R}\big(I_C J \mp J R_C J\big)\mathcal{R}^\dagger \geqslant 0 & \text{for } (F_L, f_L), \\
u_j = \mathcal{R}K_j\mathcal{R}^\dagger, \ \eta_j = \gamma_j & \text{for } (F_U, f_U).
\end{cases}
\tag{3.34}
$$

with the remaining terms vanishing.

In our Second Paper, we examine the conditions (3.33) in detail. Based on our analysis, we conclude that symplectic evolution is semiclassical (with respect to the RSF toolbox) if and only if it consists of *passive* transformations, i.e. transformations which are orthogonal in addition to being symplectic. In quantum-optical experiments, passive transformations correspond to operations which are energy-preserving and with known classical analogs, such as beam-splitters and phase-shifters [39]. According to standard notions of non-classicality, such as non-positivity of the Glauber P representation or the presence of quantum entanglement, the output of passive transformations can be non-classical only if given non-classical input [114, 115]. The remaining *active* transformations, such as squeezing, which can be a source of quantum advantage [116, 117], have no classical interpretation. Such transformations are forbidden by Eq. (3.33). These findings apply in particular to the evolution stemming from unitary Lindblad operators investigated in our First Paper: such evolution is semiclassical only if the scattering events it describes are energy-preserving.

To illustrate these results, our Second Paper provides a number of examples. Here, let us consider just one of them: *stabilizability* in two-mode entangled Gaussian systems. In quantum open systems, it is sometimes desirable to counteract the effects of dissipation by using an appropriate Hamiltonian. In the framework of stabilizability, one can check whether this is possible by solving a finite set of conditions rather than checking every Hamiltonian separately [118, 37]. Recently, one of our other papers [OP5] employed stabilizability to investigate the robustness of two-mode Gaussian states against three classes of dissipation based on linear Lindblad operators:

1. Local damping: $\hat{L}_1 := \hat{a}_1$, $\hat{L}_2 := \hat{a}_2$;

2. Global damping: $\hat{L} := (\hat{a}_1 + \hat{a}_2)$;

3. Dissipators engineered to preserve two-mode squeezed states:

$$
\begin{aligned}
\hat{L}_1 &:= \cosh\chi\,\hat{a}_1 - \sinh\chi\,\hat{a}_2^\dagger, \\
\hat{L}_2 &:= \cosh\chi\,\hat{a}_2 - \sinh\chi\,\hat{a}_1^\dagger,
\end{aligned}
\tag{3.35}
$$

where $\chi \geqslant 0$ denotes the amount of squeezing.

It is straightforward to check that while all the dissipators fulfill the relevant classicality condition (3.33), only the first two result in non-negative particle creation rates as required in Eq. (3.34). This, of course, makes sense from the point of classicality, since squeezing is a purely quantum resource, while the Lindblad operators appearing in the first two models merely describe particle loss in the system.

In addition, we remark that in the first and third models, the maximum amount of entanglement was stabilized in the system when using the Hamiltonian

$$\hat{H}_{\mathrm{sq}} := -i\omega\big(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger\big), \tag{3.36}$$

while in the second model the entanglement-maximizing Hamiltonian read as

$$\hat{H} = \hat{H}_{\mathrm{cas}} := -\frac{i\omega}{2}\left[\big(\hat{a}_1 + \hat{a}_2\big)^2 - \big(\hat{a}_1^\dagger + \hat{a}_2^\dagger\big)^2\right]. \tag{3.37}$$

As expected, neither Hamiltonian fulfills the classicality condition (3.33).

3.3   *Classicality of the Bogoliubov transformations and the dynamical Casimir effect through the reduced state of the field* [Third Paper]

Our results so far determine that symplectic evolution, as given by the quantum Gaussianity-preserving equations (2.18, 3.6), is classical if and only if it is built solely from passive operations. However, as discussed in Section 2.2, symplectic evolution can also arise from Bogoliubov transformations. How do the classicality conditions (3.33) translate to such a setting? Moreover, what can they tell us about the dynamical Casimir effect? Advancing on these question is the main goal of our Third Paper, summarized in this section.

3.3.1   Classicality of Bogoliubov transformations

To derive the classicality conditions for Bogoliubov transformations, our Third Paper follows the same methodology as our Second Paper: if the transformations preserve the set of the degrees of freedom contained within RSF, we regard them as semiclassical, and if not, we regard them as inherently quantum. Proceeding in this way, we obtain the following results (below we follow the notation established in Section 2.2):

Proposition 5. *Isolated system Bogoliubov transformations are compatible with the RSF formalism and are thus classical with respect to it if and only if*

$$0 = \mathcal{X}_\downarrow. \tag{3.38}$$

Additionally, if the transformation depends smoothly on time, the corresponding reduced kinetic equations (2.45) exist and are governed by

$$h = \frac{i}{2} \left(\frac{d\mathcal{X}_\uparrow}{dt} \mathcal{X}_\uparrow^{-1} - \mathcal{X}_\uparrow^{-\dagger} \frac{d\mathcal{X}_\uparrow^\dagger}{dt} \right). \tag{3.39}$$

with the remaining terms vanishing.

Proposition 6. *A necessary condition for open system Bogoliubov transformations to be compatible with the RSF formalism and thus be classical with respect to it is*

$$0 = \mathcal{X}_{\downarrow S}. \tag{3.40}$$

Let us briefly discuss these findings. The obtained classicality condition for the closed system is easy to interpret: substituting (3.38) into the symplectic condition (2.27), we immediately find that $\mathcal{X}$ is also unitary in additional to being symplectic, which for complex symplectic transformations corresponds to their passiveness. Thus, in a complete analogy to quantum Gaussian evolution studied in our Second Paper, Bogoliubov transformations in isolated systems are semiclassical only if they correspond to passive transformations.

The case of the open system is more subtle. When we compare Eq. (3.40) with its closed system counterpart, we can observe that the latter is considerably

more restrictive. It requires the entire matrix $\mathcal{X}_\downarrow$ to vanish, while the former only requires this from the system component $\mathcal{X}_{\downarrow S}$. Consequently, depending on how we define the degrees of freedom of the system, we might perceive the same overall dynamics as either classical or quantum from the system's perspective. Still, it is crucial to emphasize that the condition (3.40) is not equivalent to classicality: it is only necessary for it. Our Third Paper demonstrates that, in stark contrast to the closed system, the classicality of an open system Bogoliubov transformation depends not only on the transformation matrix $\mathcal{X}$, but also the entire initial system-environment ensemble. For highly correlated initial ensembles, the only semiclassical Bogoliubov transformations may happen to be only those that result in completely separate dynamics for the system and the environment, effectively defining a closed system.

To make stronger statements, we are therefore forced to make some restrictions. Firstly, we assume that the initial ensemble is separable with respect to the bipartition between the system and the environment, which is typical in the theory of quantum open systems [119]. Secondly, we assume that the bath is initially in the vacuum state. Note that, while this assumption is admittedly strong, it is fulfilled by many useful models, such as quantum limited amplification, quantum limited attenuation and phase conjugation channels, utilized, e.g. in studies of Gaussianity, entropy and entanglement [120, 121, 122]. Indeed, the example of quantum limited amplification is discussed in our Third Paper in detail. More importantly, as we will show in the next section, the vacuum state assumption is also satisfied by the dynamical Casimir effect.

Under the above assumptions, our Third Paper proves our final main result for Bogoliubov transformations:

Proposition 7. *The classicality condition (3.40) is both necessary and sufficient for open system Bogoliubov transformations with the environment initially in the vaccum state. Additionally, if such transformations depend smoothly on time, the corresponding reduced kinetic equations exist provided*

$$\mathcal{W} \geqslant 0, \quad \mathcal{W} - \mathcal{Y}_r \geqslant 0 \tag{3.41}$$

and are governed by

$$h = -\mathcal{Y}_i/2, \quad \gamma_\downarrow = \mathcal{W}, \quad \gamma_\uparrow = \mathcal{W} - \mathcal{Y}_r, \tag{3.42}$$

with the remaining terms vanishing. Here,

$$
\begin{aligned}
\mathcal{Y}_i &:= -i\left(\mathcal{Y} - \mathcal{Y}^\dagger\right), \quad \mathcal{Y} := \frac{d\mathcal{X}_{\uparrow S}}{dt}\mathcal{X}_{\uparrow S}^{-1}, \\
\mathcal{Y}_r &:= \mathcal{Y} + \mathcal{Y}^\dagger, \qquad\quad \mathcal{D} := \mathcal{X}_{\downarrow C}\mathcal{X}_{\downarrow C}^\dagger, \\
\mathcal{W} &:= \frac{d\mathcal{D}}{dt} - \mathcal{Y}\mathcal{D} - \mathcal{D}\mathcal{Y}^\dagger.
\end{aligned}
\tag{3.43}
$$

3.3.2 Classicality of the dynamical Casimir effect

Armed with Propositions 5-7, we are ready to come back to the dynamical Casimir effect. We begin by observing that, while the phenomenon spans an infinite number of modes of photons with both helicities, its defining Bogoliubov transformation (2.32) couples them in pairs only. Any mode $\vec{k}$ of the right helicity photons

is coupled only to itself and the mode $-\vec{k}$ of the left helicity photons. For this reason, we can restrict our analysis to two modes, with no loss in generality.

As is easy to observe, written in terms of the matrix $\mathcal{X}$, the Bogoliubov transformation (2.32) reads

$$\mathcal{X} = \begin{bmatrix} e^{-i\phi}f_{R+} & 0 & 0 & e^{-i\phi}f_{R-} \\ 0 & e^{i\phi}f_{L-}^* & e^{i\phi}f_{L+}^* & 0 \\ 0 & e^{i\phi}f_{R-}^* & e^{i\phi}f_{R+}^* & 0 \\ e^{-i\phi}f_{L+} & 0 & 0 & e^{-i\phi}f_{L-} \end{bmatrix}. \tag{3.44}$$

The classicality interpretation depends on what we consider to be the system. In the most natural view, the system spans photons with both left and right helicity. Hence, we have a closed, two-mode system. Comparing Eq. (3.44) with (2.28), we easily find the classicality criterion (3.38) to read explicitly

$$f_{R-} = 0 = f_{L+}. \tag{3.45}$$

Looking at Eq. (2.33), we can immediately see that this implies no Casimir effect, i.e. the photon production in the vacuum is zero. This is fully expected when we realise that, as we prove in our Third Paper, the restriction (3.45) implies constant velocity of the medium, for which, as is well-known, the Casimir effect cannot take place.

However, there is another perspective. Nothing stops us from interpreting exclusively the left helicity photons as the system, and the right helicity photons as the environment, so that the system is open. By comparing (3.44) with Eqs. (2.28, 2.30), we immediately find that now, the classicality condition (3.40) always holds, regardless of the form of the functions $f_{R\pm}$, $f_{L\pm}$. Importantly, because the mode associated with the right helicity photons, i.e. the environment, is initially in the vacuum state, then, due to Proposition 7, this classicality condition is both necessary and sufficient. From this point of view, contrary to its reputation as a radically quantum effect, the Casimir effect appears to be semiclassical.

To explain this, let us briefly investigate the maximally entangled two-qubit Bell state [123, 124]:

$$|\Phi_+\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right). \tag{3.46}$$

If we consider only the first qubit as the system, we will find it to be in the maximally mixed state

$$\hat{\rho}_S = \text{Tr}_{\text{2nd qubit}}|\Phi_+\rangle\langle\Phi_+| = \frac{1}{2}\hat{\mathbb{1}}_2, \tag{3.47}$$

which can certainly be considered classical. Of course, this does not mean that the Bell state that we started with was classical. Instead, its "quantumness" was contained in the correlations between the two qubits, rather than any of the two qubits themselves.

In complete analogy, the Casimir effect itself is not classical – only its appearance from the point of view of the system is. Consider the matrix element $\mathcal{X}_{\downarrow 12} = \mathcal{X}_{\downarrow C} = f_{R-}$, which in our case encodes the correlations between photons with left and right helicities. For a generic initial state, these correlations are

potentially quantum. Thus, if a closed system is to be considered classical, they must necessarily vanish: $\mathcal{X}_{\downarrow 12} = \mathcal{X}_{\downarrow C} = 0$, as they constitute an integral part of the system. However, from the point of view of an open system, the discussed correlations enter it only at the level of the environmental effects, through Eq. (3.43). Therefore, even if they have a strictly quantum origin, the system experiences them only as dissipation, which in this case happens to have a semiclassical interpretation in terms of particle annihilation and creation rates.

As the penultimate result, our Third Paper considers Eq. (2.32) as defining a smooth Bogoliubov evolution in the parameter T. Using our recipe (3.42), we provide the explicit formulas for the corresponding components in the reduced kinetic equations (2.45). Notably, we find that the Hamiltonian for the photons is proportional to their frequency, while the particle annihilation rate is zero, which intuitively corresponds to the fact that the dynamical Casimir effect results only in the spontaneous creation of particles, not their disappearance.

Finally, we calculate that the time derivative of the photon densities (2.33) equals

$$\frac{d}{dT}\left\langle \hat{n}_{L/R} \right\rangle = \gamma_{\uparrow}\left(\left\langle \hat{n}_{L/R} \right\rangle + 1 \right). \tag{3.48}$$

As seen, the non-negativity of $\gamma_{\uparrow}$, which is required for the result to be physical, is equivalent to the non-negativity of photon number growth. Furthermore, because of the $\langle \hat{n}_{L/R} \rangle$-independent term on the r.h.s., the equation proves that the dynamical Casimir effect occurs for any non-zero $\gamma_{\uparrow}$, which can be traced to any non-zero acceleration of the medium.

Chapter 4

Outlooks

The quantum-to-classical transition, being of both practical and conceptual significance, remains one of the key issues in contemporary theoretical physics. In our dissertation, we approached the problem for the specific case of so-called symplectic evolution, which governs the dynamics of most forms of light accessible in modern quantum optical experiments. Most notably, we found that such evolution can be considered semiclassical if and only if it is built upon passive transformations, which possess well-established classical interpretation. Our findings are based on the recent mesoscopic formalism of the reduced state of the field (RSF), originally a framework for macroscopic bosonic fields, which we redeveloped as a tool for probing classicality.

Although we achieved the planned goals, there is still much to discover when it comes to classicality within quantum mechanics. To start with, some of our results could be generalized. For example, can our conjecture that RSF contains no information about any form of quantum entanglement be proven? Similarly, is there a way to strengthen our conditions for classicality of Bogoliubov transformations, so that they are sufficient for a broad class of environments, such as, e.g. arbitrary thermal baths? Could they account for any well-known phenomena based on Bogoliubov transformations beyond the dynamical Casimir effect, e.g. the Unruh effect? Going further, both of the mesoscopic formalisms studied by us (symplectic picture and RSF) are built upon one- and two-point correlation functions. Can a self-consistent mesoscopic framework based on higher-order correlations be designed? If so, what new insights could it offer, in particular with respect to classicality?

Last but not least, potentially novel considerations may follow from our proposed addition to symplectic evolution based on unitary Lindblad operators. Most obviously, as shown by us, the dynamics could be employed to account for the influence of noise in the form of random scattering in standard Gaussian dynamics, or to drive a system towards a desired resourceful state. Moreover, symplectic evolution itself could still be extended: although operations preserving the set of Gaussian states are fully characterized [125], an analogous problem was not solved for quantum Gaussian states. It would be interesting to see what other evolution models preserve the set of quantum Gaussian states and the symplectic description, as well as what physical phenomena they relate to.

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Dissipative evolution of quantum Gaussian states

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Recent works on quantum resource theories of non-Gaussianity, which are based upon the type of tools available in contemporary experimental settings, put Gaussian states and their convex combinations on equal footing. Motivated by this, in this paper, we derive a model of dissipative time evolution based on unitary Lindblad operators which, while it does not preserve the set of Gaussian states, preserves the set of their convex combinations, i.e., so-called quantum Gaussian states. As we demonstrate, the considered evolution proves useful both as a description for random scattering and as a tool in dissipator engineering.

I. INTRODUCTION

One of the most prominent families of states in continuous variable quantum mechanics consists of Gaussian states, i.e., states with Gaussian (normal) characteristic functions. Due to their relative simplicity in both analytical description and practical implementation, they found extensive use in fields as varied as quantum optics, information and thermodynamics, among others [1–4].

Consequently, much interest was devoted to time evolution which preserves the set of Gaussian states. Such evolution is described by the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) equation generated by a polynomial of at most second degree in the quadrature operators. Over the years, it proved to be successful in studies of quantum thermodynamics [5], optics [6], entanglement [7,8], discord [9], purity [10,11], fidelity [12], steering [13], stabilizability [14,15] and classical limits of quantum mechanics [16,17], among others.

Despite the popularity of Gaussian states and dynamics, in comparison, little interest was devoted to the so-called quantum Gaussian states, which are a generalization of Gaussian states that also includes their convex combinations [18–23]. However, according to recent developments in quantum resource theories of non-Gaussianity [24,25], which are motivated by the type of operations available in modern experiments employing continuous variables, Gaussian states and their convex combinations are equally resourceful. From this perspective, the aforementioned restriction to evolution preserving the set of Gaussian states is too severe and should be relaxed to allow the more general quantum Gaussian states.

In this paper, we develop an explicit model of time evolution compatible only with this weaker restriction: it preserves the convex hull of Gaussian states and not the Gaussian family of states itself. Despite that, it is fully compatible with the symplectic (covariance matrix) picture used extensively in studies of Gaussian phenomena. The model is derived from the central assumption of unitary Lindblad operators, a class studied first in 1972 within the then-rapidly developing field of quantum dynamical semigroups [26–28].

The considered evolution has two very different applications depending on the nature of unitary Lindblad operators entering it. For a large number of noncommuting operators, we use a combination of the collision model and kicked top dynamics to show that the evolution describes random scattering, a view consistent with the first findings regarding unitary Lindblad operators [26]. On the other hand, for a single Lindblad operator, time evolution may be employed in dissipator engineering, which we demonstrate with an example of entanglement creation in two-mode states.

The paper is organized as follows. Section II is devoted to preliminaries: a symplectic picture of quantum states and evolution that preserves the set of Gaussian states. In Sec. III, we develop the discussed evolution equation and study its basic technical properties: Gaussianity and symplectic representation. In Sec. IV, we consider the evolution as a description of random scattering. In Sec. V, we investigate the stationary solutions of the derived evolution equation, which we then use in Sec. VI in an engineered dissipation scenario for entanglement harvest. We conclude in Sec. VII.

II. SYMPLECTIC PICTURE

Studies of Gaussian states and their evolution often make use of the symplectic picture, which reduces the N-mode infinitely dimensional Hilbert space associated with the density operator to a space of dimension $2N$, which is typically easier to work with. Here, we briefly summarize the relevant information about the symplectic picture, including the so-called covariance matrix and Gaussianity-preserving evolution.

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A. Covariance matrix and vector of means

Let us consider an N-mode Hilbert space $\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k$ equipped with N pairs of quadrature operators $\hat{x}_k$, $\hat{p}_k$, conveniently collected in a single vector:

$$\hat{\vec{\xi}} := (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_N, \hat{p}_N)^T. \tag{1}$$

As the quadrature operators form a basis of operators acting on $\mathcal{H}$, every state describing the system can be fully characterized [29] by the complete ($n = 1, \dots, \infty$) set of nth-*order correlation functions (correlations)* of the form

$$\langle \hat{\xi}_{l_1} \dots \hat{\xi}_{l_n} \rangle := \text{Tr}\big[\hat{\rho}\, \hat{\xi}_{l_1} \dots \hat{\xi}_{l_n}\big], \tag{2}$$

which we also call nth *moments* for short. In many studies, especially those involving Gaussian states, i.e., states with Gaussian characteristic functions [1,30,31], it is enough to consider only the first and second moments. The advantage is that, in contrast to the infinitely dimensional density operator, the first two moments are completely described by a moderate number of degrees of freedom [32].

Information about the first moments is contained in a $2N$-dimensional *vector of means*

$$\xi_k := \langle \hat{\xi}_k \rangle, \tag{3}$$

while the second moments are encoded in the $2N \times 2N$ *covariance matrix*

$$V_{kk'} := \frac{1}{2}\langle \{\hat{\xi}_k, \hat{\xi}_{k'}\} \rangle - \xi_k \xi_{k'}. \tag{4}$$

Both $\{\cdot, \cdot\}$ and $[\cdot, \cdot]$ as usual denote commutators and anticommutators, respectively. Any valid covariance matrix has to be positive and fulfill the Heisenberg uncertainty relations (we assume natural units):

$$\sqrt{\langle \hat{x}_k^2 \rangle - \langle \hat{x}_k \rangle^2}\sqrt{\langle \hat{p}_k^2 \rangle - \langle \hat{p}_k \rangle^2} \geqslant \frac{1}{2}, \tag{5}$$

where $k \in \{1, \dots, N\}$, equivalent to [30]

$$V + \frac{i}{2}J \geqslant 0. \tag{6}$$

Here, J is the *symplectic form*, defined in terms of the canonical commutation relations as

$$J_{kk'} := -i[\hat{\xi}_k, \hat{\xi}_{k'}], \tag{7}$$

and explicitly equal to

$$J = \bigoplus_{k=1}^{N} J_2, \quad J_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{8}$$

The symplectic form defines the symplectic group $Sp(2N, \mathbb{R})$ consisting of matrices K of size $2N \times 2N$, such that [33]

$$KJK^T = J. \tag{9}$$

In this paper, special emphasis is put on a subset of symplectic matrices which possess the following exponential representations (both of which are useful depending on the context) [34,35]:

$$K = e^{JS} \equiv e^{S'J}, \tag{10}$$

for some symmetric matrices S and $S' = JSJ^T$. We stress that while all matrices of the form (10) are symplectic [36], not all

symplectic matrices are of this form due to the fact that the symplectic group is not compact [37,38].

The pair $(V, \vec{\xi})$ defines the *symplectic picture* (also known as the *covariance matrix picture*) of quantum states. All standard notions known from the density operator picture translate in a natural way to the symplectic picture. In particular, just like any density operator can be diagonalized by a unitary operation and is therefore described by its eigenvalues, any covariance matrix can be brought to a diagonal form by a symplectic operation and is described by its symplectic eigenvalues:

$$1/2 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_N. \tag{11}$$

The symplectic eigenvalues come in pairs, i.e., the diagonalized covariance matrix reads $V_{\text{diag}} = \text{diag}(\nu_1, \nu_1, \dots, \nu_N, \nu_N)$. Furthermore, they are related to the eigenvalues μ_j of the matrix JV via

$$i\mu_{2k} = -i\mu_{2k-1}^* = \nu_k, \quad k \in \{1, \dots, N\}. \tag{12}$$

In the case of Gaussian states, the symplectic picture is complete, i.e., it is equivalent to the density operator description. Otherwise, it describes a subset of the system's degrees of freedom.

B. Gaussianity-preserving evolution

In the theory of quantum dynamical semigroups, the state of the system at time $t \geqslant 0$ is given by

$$\hat{\rho}(t) = e^{t\mathcal{L}\cdot}\hat{\rho}(0). \tag{13}$$

Here, $\mathcal{L}$ is the *generator of evolution*, which has the general form

$$\mathcal{L}\cdot = -i[\hat{H}, \cdot] + \sum_j \big(\hat{L}_j \cdot \hat{L}_j^\dagger - \tfrac{1}{2}\{\hat{L}_j^\dagger \hat{L}_j, \cdot\}\big). \tag{14}$$

The system Hamiltonian $\hat{H}$ is responsible for unitary evolution, while the Lindblad operators (Lindbladians) $\hat{L}_j$ govern the dissipative part of the dynamics.

Here and below we use the dot to denote the argument of the generator, e.g., the action $\mathcal{L}\hat{\rho}$ of the generator on a generic state is given by the right-hand side of Eq. (14) with the dot replaced by $\hat{\rho}$. On the other hand, the exponential of the generator is to be understood in terms of its repeated application on the state via

$$e^{t\mathcal{L}\cdot}\hat{\rho} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\mathcal{L}\mathcal{L}\dots\mathcal{L}}_{n \text{ times}}\hat{\rho}. \tag{15}$$

This convention is followed by us throughout the paper.

By differentiating both sides of Eq. (13) with respect to time, we obtain the *GKLS (Lindblad) equation* [39–41]:

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_j \big(\hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \tfrac{1}{2}\{\hat{L}_j^\dagger \hat{L}_j, \hat{\rho}\}\big). \tag{16}$$

If the generator is a polynomial of at most second degree in the quadrature operators, the evolution preserves the set of Gaussian states. In such cases, the Hamiltonian equals

$$\hat{H} = \tfrac{1}{2}\hat{\vec{\xi}}^T G \hat{\vec{\xi}}, \tag{17}$$

where G is a $2N \times 2N$ real, symmetric matrix. The Lindblad operators, on the other hand, equal

$$\hat{L}_j = \sum_{k=1}^{2N} (\vec{c}_j)_k \hat{\xi}_k, \quad \vec{c}_j \in \mathbb{C}^{2N}, \tag{18}$$

necessarily being just linear in the quadratures.

Computing the time derivative of the covariance matrix and assuming that the system evolves according to the GKLS equation specified by Eqs. (17) and (18), we obtain the corresponding equations for the covariance matrix and the vector of means [7,10,14,15]:

$$\begin{aligned} \frac{d}{dt} V &= AV + VA^T + J\mathrm{re}C^\dagger CJ^T, \\ \frac{d}{dt}\vec{\xi} &= A\vec{\xi}, \end{aligned} \tag{19}$$

where $A := J[G + \mathrm{im}C^\dagger C]$ and $C_{jk} := (\vec{c}_j)_k$.

III. DISSIPATIVE EVOLUTION STEMMING FROM UNITARY LINDBLAD OPERATORS

Quantum resource theories classify quantum operations and states according to a given physical property, typically corresponding to usefulness with respect to some practical tasks [42]. For example, in resource theories of entanglement, entangled states are considered resourceful, while separable states are classified as free [43,44]. Accordingly, operations incapable of creating entangled states from separable ones are also deemed free. Such classification is natural from the experimental point of view, since, like any valuable resource, entanglement is useful yet difficult to obtain, while operations preserving the set of separable states are relatively easy to implement. By calling entanglement a resource, one can better pose and answer practical questions; e.g., assuming no limits on free operations, how much entanglement is needed to realize a given teleportation protocol?

In the resource theories of Gaussianity [24,25], the set of free operations consists of operations routinely available in current experiments employing continuous variable quantum systems. These include Gaussianity-preserving unitary operations, compositions with Gaussian states, and homodyne measurements. In such a setting, the emergent free states (which are preserved by the free operations) are *quantum Gaussian*, that is, they consist of Gaussian states and their convex combinations [20,21,23] (we stress that quantum Gaussian states and Gaussian states are not the same, as the former generalize the latter). From this resource-theoretic point of view, it is natural to look for physically meaningful evolution preserving the set of quantum Gaussian states. By definition, such evolution requires no input resources and can be thus used to manipulate a given system at no cost.

Observe that the usually assumed Gaussian dynamics (19) already preserve the set of quantum Gaussian states: since they map Gaussian states to Gaussian states, then, by linearity, they also map their convex combinations to other such combinations. Thus, the generator of Gaussian dynamics, given by Eq. (14) with Eqs. (17) and (18) at the input, preserves the set of quantum Gaussian states. However, in principle, there may exist other generators that preserve the set of quantum Gaus-

sian states without necessarily preserving the set of Gaussian states. This is exactly what we investigate here.

A. The model of time evolution

Let us go back to the GKLS equation (16). Being interested in the dissipative part of the equation only, we can disregard the Hamiltonian term. As for the dissipator, we follow [26–28,45,46] and consider a particular case of M Lindblad operators, all being proportional to unitary operators:

$$\hat{L}_j = \sqrt{\gamma_j}\hat{U}_j, \tag{20}$$

where $\gamma_j \geqslant 0$, $\hat{U}_j\hat{U}_j^\dagger = \hat{U}_j^\dagger\hat{U}_j = \hat{\mathbb{1}}$, and $\hat{U}_j$ is moreover assumed to be Gaussianity preserving. All our results are based on this central assumption. The corresponding GKLS equation is generated by

$$\mathcal{L}\cdot = \sum_{j=1}^{M} \gamma_j(\hat{U}_j \cdot \hat{U}_j^\dagger - \hat{\mathbb{1}}) \tag{21}$$

and thus reads

$$\frac{d}{dt}\hat{\rho} = \sum_{j=1}^{M} \gamma_j(\hat{U}_j\hat{\rho}\hat{U}_j^\dagger - \hat{\rho}). \tag{22}$$

For convenience, we assume that γ_j fulfill $\sum_{j=1}^{M} \gamma_j = 1$.

We stress that the choice (20) of Lindblad operators constitutes a certain loss of generality with respect to the general GKLS equation (16). For example, in the considered case, operators $\hat{L}_j^\dagger\hat{L}_j$ and $\hat{L}_j\hat{L}_j^\dagger$ are proportional to the identity, and consequently both commute with any state $\hat{\rho}$. Such property is not fulfilled by generic Lindblad operators.

To see that Eq. (22) preserves the set of quantum Gaussian states, we start with a single unitary operation. Since Eq. (22) is a subclass of the GKLS evolution, its formal solution is given by Eq. (13) with generator (21). For a single Lindbladian, the latter reduces to $\mathcal{L}\cdot = \hat{U}\cdot\hat{U}^\dagger - \hat{\mathbb{1}}$. The identity commutes with any operator, so

$$\hat{\rho}(t) = e^{t\hat{U}\cdot\hat{U}^\dagger}e^{-t\hat{\mathbb{1}}}\hat{\rho}(0) = \sum_{k=0}^{\infty} p_k(t)\hat{U}^k\hat{\rho}(0)(\hat{U}^\dagger)^k, \tag{23}$$

where

$$p_k(t) := e^{-t}t^k/k! \tag{24}$$

is the Poisson distribution.

Similarly, for an arbitrary number of Lindbladians, we have

$$\hat{\rho}(t) = \sum_{k=0}^{\infty}\sum_{l_1...l_k=1}^{M} p_{l_1...l_k}(t)\hat{U}_{l_k}...\hat{U}_{l_1}\hat{\rho}(0)\hat{U}_{l_1}^\dagger...\hat{U}_{l_k}^\dagger, \tag{25}$$

where for $k = 0$ the summand is $e^{-t}\hat{\rho}(0)$ and for $k > 0$ $p_{l_1...l_k}(t) := \gamma_{l_1}...\gamma_{l_k}e^{-t}t^k/k!$.

Any unitary operator has an exponential representation of the form

$$\hat{U}_j = e^{-i\hat{h}_j}, \tag{26}$$

for some Hermitian operator $\hat{h}_j$, called the operator's *generator* [not to be confused with the generator of the GKLS evolution (14)]. As is well known [24,25,31], unitary operations with generators that are polynomials of at most second

degree in quadrature operators preserve the set of Gaussian states. Furthermore, if each $\hat{U}_{l_j}$ preserves Gaussian states, then so does $\hat{U}_{l_k} \ldots \hat{U}_{l_1}$, and therefore each of the terms in the sum (25) maps Gaussian states to other Gaussian states.

Since the sum of Gaussian states is in general not Gaussian, then even for an initial Gaussian state the time-evolved state (23) is also not Gaussian in general. On the other hand, if the initial state is a convex combination of Gaussian states, then, by linearity, the time-evolved state is also a convex combination of Gaussian states. Therefore, under the assumption that each Lindblad operator (20) is generated by a polynomial of at most second degree in quadrature operators, Eq. (32) preserves the set of quantum Gaussian states without preserving the set of Gaussian states, as we wanted to show.

B. Representation in the symplectic picture

One of the advantages of working with Gaussian states is that Gaussianity-preserving evolution corresponds to self-contained Eqs. (19) in the symplectic picture, by which we mean that the evolution of the covariance matrix and the vector of means can be traced without having to consider third- and higher-order correlation functions. As we show here, this property extends to Eq. (22), allowing one to study the evolution of quantum Gaussian states in the same fashion as in the case of Gaussian states.

Multiplying Eq. (22) by appropriate polynomials in the quadrature operators and taking the trace, we obtain the corresponding evolution of the first and second moments:

$$
\frac{d}{dt}\langle \hat{\xi}_n \hat{\xi}_{n'} \rangle = \sum_{j=1}^{M} \gamma_j \langle \hat{\xi}_{n,j} \hat{\xi}_{n',j} - \hat{\xi}_n \hat{\xi}_{n'} \rangle,
$$

$$
\frac{d}{dt}\langle \hat{\xi}_n \rangle = \sum_{j=1}^{M} \gamma_j \langle \hat{\xi}_{n,j} - \hat{\xi}_n \rangle,
$$

(27)

where

$$
\hat{\xi}_{n,j} := \hat{U}_j^\dagger \hat{\xi}_n \hat{U}_j
$$

(28)

denotes transformed quadrature operators.

Clearly, if the transformed quadrature operators are linear in the initial quadratures, then Eqs. (27) are closed with respect to the first two moments. In order for the new quadratures to have a physical meaning, they should also fulfill the canonical commutation relations. A generic transformation fulfilling these conditions is called a *Bogoliubov transformation* [47–49]. In the case at hand, a generic Bogoliubov transformation reads explicitly

$$
\hat{\xi}_{n,j} = \sum_{m=1}^{2N} (K_j)_{nm} \hat{\xi}_m,
$$

(29)

where K_j is a real symplectic matrix of size $2N \times 2N$. Under the assumption that the Lindbladians (26) are generated by polynomials of at most second degree in quadrature operators, the matrices K_j possess the convenient exponential representation (10).

Taking the time derivative of the covariance matrix and the vector of means with Eqs. (27)–(29) at the input yields the

symplectic picture equivalent to Eq. (22):

$$
\frac{d}{dt}V = \sum_{j=1}^{M} \gamma_j \left[K_j V K_j^T - V + F_j(\vec{\xi}) \right],
$$

$$
\frac{d}{dt}\vec{\xi} = \sum_{j=1}^{M} \gamma_j K_j \vec{\xi},
$$

(30)

where

$$
F_j(\vec{\xi}) = (K_j - \mathbb{1})\vec{\xi}\,\vec{\xi}^T (K_j^T - \mathbb{1}).
$$

(31)

Note that in typical applications of the covariance matrix evolution, concerning, e.g., quantum entanglement, the vector of mean values is irrelevant. For this reason, later on we will assume $\vec{\xi}(0) = 0$, in which case $F_j(\vec{\xi}) = 0$ [50] and the evolution simplifies to

$$
\frac{d}{dt}V = \sum_{j=1}^{M} \gamma_j \left[K_j V K_j^T - V \right].
$$

(32)

The corresponding explicit solutions are

$$
V(t) = \sum_{j=0}^{\infty} p_j(t) K^j V(0) (K^T)^j
$$

(33)

for a single Lindbladian and

$$
V(t) = \sum_{k=0}^{\infty} \sum_{l_1 \ldots l_j = 1}^{M} p_{l_1 \ldots l_j}(t) K_{l_j} \ldots K_{l_1} V(0) K_{l_1}^T \ldots K_{l_j}^T
$$

(34)

for an arbitrary number of Lindbladians.

As we investigate below, depending on the number and nature of the unitary Lindblad operators, Eq. (22) and its symplectic representation (32) can have radically different applications, ranging from random scattering to engineered dissipation.

IV. RANDOM SCATTERING

We now employ the collision model and kicked top dynamics to show that for a large number M of noncommuting Lindblad operators, the discussed evolution constitutes a natural description of random scattering.

A. Derivation from the collision model

In the collision model [51–53], the initial system is coupled to an infinite number of identical copies of ancilla $\hat{\eta}$. The total initial state is separable:

$$
\hat{\rho}_T(0) = \hat{\rho}(0) \otimes \hat{\eta} \otimes \hat{\eta} \otimes \ldots .
$$

(35)

During the first time step Δt, a unitary operation $\hat{W}_1$ acts on the system and the first ancilla, after which the latter is traced out. The resulting state of the system is thus

$$
\hat{\rho}(\Delta t) = \mathrm{Tr}_\eta \{ \hat{W}_1 [\hat{\rho}(0) \otimes \hat{\eta}] \hat{W}_1^\dagger \},
$$

(36)

where Tr_η denotes the partial trace over the ancilla. Since the corresponding total state has the same form as initially (35),

$$
\hat{\rho}_T(\Delta t) = \hat{\rho}(\Delta t) \otimes \hat{\eta} \otimes \hat{\eta} \otimes \ldots ,
$$

(37)

the second and further steps lead to analogous results as the first one. After n steps

$$\hat{\rho}(n\Delta t) = \mathrm{Tr}_\eta\{\hat{W}_n[\hat{\rho}[(n-1)\Delta t] \otimes \hat{\eta}]\hat{W}_n^\dagger\}. \quad (38)$$

The unitaries $\hat{W}_n$ are typically assumed to have the elementary form [53]

$$\hat{W}_n = \exp\left[-i(\hat{w}_S + \hat{w}_\eta + \hat{w}_{\mathrm{int},n})\Delta t\right], \quad (39)$$

where the Hamiltonian $\hat{w}_S$ acts on the system, $\hat{w}_\eta$ acts on the bath, while $\hat{w}_{\mathrm{int},n}$ is responsible for interaction between the two. The last Hamiltonian may be step dependent, while the others are assumed to be the same in each step. All three operators are time independent.

Here, we employ a more general model [52], in which the unitary operators are unrestricted. This gives the following general form:

$$\hat{W}_n = \mathcal{T}\exp\left(-i\int_{(n-1)\Delta t}^{n\Delta t} d\tau\ \hat{w}_n(\tau)\right) \quad (40)$$

where $\mathcal{T}$ is the time-ordering operator and the time-dependent Hamiltonian $\hat{w}_n$ can act on both the system and the nth ancilla in an arbitrary way.

Clearly, by choosing the ancilla and the unitaries accordingly, we can use the collision model to emulate a wide range of dynamics. This fact, coupled with the relative conceptual simplicity, makes the collision model a popular tool in dealing with topics as varied as optics, thermodynamics, and non-Markovianity, among others [52,53]. Here, we use the collision model framework to derive the GKLS equation with M unitary Lindblad operators.

For the ancillas, we choose qudits of dimension $d = M + 1$ in the ground state:

$$\hat{\eta} = |0\rangle\langle 0|. \quad (41)$$

Furthermore, we choose unitary operations of the form

$$\hat{W}_n = \left(\hat{\mathbb{1}} \otimes |0\rangle\langle 0| + \sum_{j=1}^{M} \hat{U}_j \otimes |j\rangle\langle j|\right)[\hat{\mathbb{1}} \otimes \hat{O}(\Delta t)], \quad (42)$$

where $\hat{U}_j$ are arbitrary unitary operators with generators $\hat{h}_j$ [which, in the case of evolution preserving the convex hull of Gaussian states, are polynomials of at most second degree in quadrature operators] and $\hat{O}$ is a time-dependent unitary matrix defined by its action on the ancilla:

$$\hat{O}(\Delta t)|0\rangle = \sqrt{1 - \Delta t}|0\rangle + \sqrt{\Delta t}\sum_{j=1}^{M}\sqrt{\gamma_j}|j\rangle. \quad (43)$$

As before, $\gamma_j \geqslant 0$ and $\sum_{j=1}^{M}\gamma_j = 1$. With these inputs, Eq. (36) becomes

$$\hat{\rho}(\Delta t) = \left[\hat{\mathbb{1}} + \Delta t\sum_{j=1}^{M}\gamma_j(\hat{U}_j \cdot \hat{U}_j^\dagger - \hat{\mathbb{1}})\right]\hat{\rho}(0). \quad (44)$$

Since in this setting we can easily recognize that $\hat{W}_n$ does not depend on the step number, each step corresponds to the same

transformation. For $t = n\Delta t$ we therefore obtain

$$\hat{\rho}(t) = \left[\hat{\mathbb{1}} + \frac{t}{n}\sum_{j=1}^{M}\gamma_j(\hat{U}_j \cdot \hat{U}_j^\dagger - \hat{\mathbb{1}})\right]^n\hat{\rho}(0). \quad (45)$$

In the continuous time limit $\Delta t \to 0$ taken simultaneously with $n \to \infty$, so that we approach a fixed value of time parameter $n\Delta t = t = \mathrm{const}$, we obtain the formal solution (13) to the GKLS equation with generator (21). In other words, we recover the solution to the GKLS equation with M unitary Lindbladians, as intended.

B. Kicked top and scattering

We demonstrated that the GKLS evolution with unitary Lindbladians can be cast into the framework of collision models. To better understand implications of this fact, we now more deeply investigate the operator $\hat{W}_n$. As seen from Eq. (42), it is an unusual product of two subunitaries: a standard unitary operator and a time-independent "kick."

Such structure is a staple in the kicked top model [54–56], defined by Hamiltonians of the form

$$\hat{H}_{\mathrm{KT}}(t) = \hat{H}_0(t) + \sum_m \delta(t - mT)\hat{V}. \quad (46)$$

Here, the standard unitary dynamics generated by the base Hamiltonian $\hat{H}_0$ are periodically disturbed (with period length T) by the delta potential $\hat{V}$, leading to chaotic behavior. Note that typically, the base Hamiltonian is assumed to be time independent. However, the results remain qualitatively the same as long as the time dependence of the Hamiltonian is well behaved (i.e., not unbounded and discontinuous like the Dirac delta distribution). Due to its relative simplicity and ease of implementation in terms of qubits, the kicked top is the theoretical [56] and experimental [57] go-to model for testing the implications of dynamical chaos on quantum phenomena (such as, e.g., entanglement).

In the Appendix, we show that the unitary operator (42) can be obtained from the general Eq. (40) by the kicked top Hamiltonian $\hat{w}_n = \hat{H}_{\mathrm{KT}}$ with

$$T = \Delta t, \quad \hat{H}_0(t) = \hat{o}_n(t), \quad \hat{V} = \sum_{j=1}^{M}\hat{h}_j \otimes |j\rangle\langle j|, \quad (47)$$

where $\hat{o}_n$ is the generator of $\hat{O}$ in the nth step [see Eq. (A4) in the Appendix for definition]. Note that, because $\hat{w}_n$ acts only during the time interval $((n-1)\Delta t, n\Delta t]$, effectively only the nth term in the sum (46) contributes.

Due to its close association with Poisson distribution [27,28], which describes random scattering through the Poisson scatter theorem [58], the GKLS equation with unitary Lindbladians constitutes a valid model of random scattering. Our results make this interpretation explicit.

Each collision can be seen as a single scattering event in the medium described by ancillas in the state (43). Crucially, the probability that the system will be kicked by the jth Hamiltonian $\hat{h}_j$ depends on γ_j through Eqs. (43) and (47). For a single Lindbladian, the system experiences identical scattering at every instant, quickly driving it towards a well-controlled stationary state (we investigate this in detail in the next section).

However, as the number of mutually noncommuting unitaries entering the equation grows, so does the uncertainty in the outcome state. In particular, in the limit $M \to \infty$ the outcome probabilities γ_j may be replaced by a probability measure $\mu(dU)$ on the unitary group, yielding a scattering integral [26]

$$\frac{d}{dt}\hat{\rho} = \int d\mu(U)(\hat{U}\hat{\rho}\hat{U}^\dagger - \hat{\rho}). \tag{48}$$

These results are consistent with previous findings [28,59] that unitary Lindbladians can be interpreted as the S matrices of the system interacting with a dilute gas.

The use of the collision model is particularly appealing also when it comes to interpreting the role of the environment. Because the ancillas are traced out after each collision, during each step the system interacts with the same bath, fulfilling the expectation that the bath should not be influenced by the scattering (in particular, future scattering should not depend on previous events).

As a final remark, we note the the notion of quantum Gaussianity was born largely to address the fact that even though convex combinations of Gaussian states are technically not Gaussian, i.e., they have non-Gaussian characteristic functions, they can be experimentally created and manipulated using the same methods as Gaussian states [24,25]. This makes Gaussian states and their convex combinations similar in practical applications. Our findings put the two families even closer, showing that states from the latter can be obtained from the former by simply subjecting them to random scattering, which may be regarded as pure noise. This result supports the developments made over the last decade to construct measures of quantum non-Gaussianity [18,19,22], which, contrary to measures of non-Gaussianity [60–64], do not assign positive values of the resource to convex combinations of Gaussian states.

V. EXPLICIT SOLUTIONS AND STATIONARY STATES

As seen, for a large number of unitary Lindblad operators, the considered evolution equation describes random scattering. However, the same equation equipped with a single Lindblad operator has well-controlled stationary states, as we proceed to show.

We start by deriving explicit solutions to the considered equation. Looking at Eq. (22), we can easily see that the stationary solutions $\hat{\rho}_\infty$ must commute with all the generators:

$$0 = [\hat{h}_j, \hat{\rho}_\infty] \quad \text{for all } j. \tag{49}$$

As the number of noncommuting Lindbladians, and thus generators, approaches infinity, the evolution begins to describe pure decoherence, driving any initial state towards the maximally mixed state in the asymptotic time limit. This view was explored by us in the previous section. However, from the point of view of engineered dissipation, we expect only a few or even a single Lindbladian to appear, in which case it is possible to drive the system towards more useful stationary solutions.

Let us thus assume a single unitary Lindbladian generated by a Hermitian operator $\hat{h}$ with eigendecomposition

$$\hat{h}|h_k\rangle = h_k|h_k\rangle, \tag{50}$$

where $h_k \in \mathbb{R}$ are assumed to be nondegenerate for simplicity. Since $\hat{h}$ is Hermitian, its eigenvectors form a basis of the Hilbert space. In particular, one can write the initial density operator in this basis:

$$\hat{\rho} = \sum_{k,k'} \rho_{kk'}^h |h_k\rangle\langle h_{k'}|. \tag{51}$$

Upon substituting into Eq. (22), we obtain

$$\frac{d}{dt}\rho_{kk'}^h = (e^{-i(h_k - h_{k'})} - 1)\rho_{kk'}^h. \tag{52}$$

This differential equation is easy to solve, yielding, after simplification, the general solution:

$$\rho_{kk'}^h(t) = e^{[\cos(h_k - h_{k'}) - 1]t} e^{-i[\sin(h_k - h_{k'})]t} \rho_{kk'}^h(0), \tag{53}$$

where $\rho_{kk'}^h(0)$ are the matrix elements of the initial state.

The stationary states follow by taking the limit $t \to \infty$. All but the diagonal terms decay exponentially, leaving

$$\hat{\rho}_\infty = \lim_{t \to \infty} \hat{\rho}(t) = \sum_k \lambda_k |h_k\rangle\langle h_k| \tag{54}$$

with the final state's eigenvalues equal to

$$\lambda_k = \rho_{kk}^h(0) = \langle h_k|\hat{\rho}(0)|h_k\rangle. \tag{55}$$

A similar results holds in the symplectic picture. It is easy to show by using Eqs. (9) and (10) that, since $JK_j = Je^{S_j J} = e^{JS_j}J$, Eq. (32) is equivalent to

$$\frac{d}{dt}(JV) = \sum_j [e^{JS_j}(JV)e^{-JS_j} - JV]. \tag{56}$$

Clearly, the stationary solutions V_∞ are given by

$$0 = [JS_j, JV_\infty] \quad \text{for all } j. \tag{57}$$

Therefore, just like in the standard picture the stationary solutions commute with the Hermitian generators of evolution, in the symplectic picture the stationary solutions "commute" (commute after multiplication by J) with the symmetric generators of evolution.

Like before, let us consider a single unitary Lindbladian, which corresponds to a single symplectic operator. We denote the eigendecomposition of JS by

$$JS\vec{s}_k = s_k\vec{s}_k. \tag{58}$$

Contrary to the Hermitian generator $\hat{h}$ from the density operator picture, JS does not have to be a normal matrix, meaning that its eigenvectors may not form a basis of the corresponding vector space. To solve the evolution equation explicitly, we consider the special case in which JS is normal. This allows us to follow the reasoning from the density operator picture.

We start by writing the initial covariance matrix as

$$JV = \sum_{k,k'} (JV)_{kk'}^s \vec{s}_k \vec{s}_{k'}^\dagger. \tag{59}$$

Upon substituting into Eq. (56), we have

$$\frac{d}{dt}(JV)_{kk'}^s = (e^{s_k - s_{k'}} - 1)(JV)_{kk'}^s, \tag{60}$$

which is solved by

$$(JV)_{kk'}^s(t) = \exp\left[(e^{s_k - s_{k'}} - 1)t\right](JV)_{kk'}^s(0). \tag{61}$$

The asymptotic time limit depends on s_k. Denoting

$$
\begin{aligned}
x_{kk'} &:= \mathrm{re}(s_k - s_{k'}), \\
y_{kk'} &:= \mathrm{im}(s_k - s_{k'}), \\
\zeta_{kk'} &:= \exp(x_{kk'})\cos(y_{kk'}),
\end{aligned} \tag{62}
$$

we obtain, as $t \to \infty$,

$$(JV)_{kk'}^s(t) \to \begin{cases} \infty & \zeta_{kk'} > 1, \\ \exp[i\tan(y_{kk'})t](JV)_{kk'}^s(0) & \zeta_{kk'} = 1, \\ 0 & \zeta_{kk'} < 1. \end{cases} \tag{63}$$

In the particular case of nondegenerate and purely imaginary s_k (the latter happens whenever K is passive, i.e., it is orthogonal in addition to being symplectic), the diagonal matrix elements approach the middle line (with $y_{kk} = 0$), while the remaining elements approach zero. Consequently, the covariance matrix approaches the stationary solution

$$JV_\infty = \lim_{t \to \infty} JV(t) = \sum_k \mu_k \vec{s}_k \vec{s}_k^\dagger \tag{64}$$

with eigenvalues

$$\mu_k = (JV)_{kk}^s(0) = \vec{s}_k^\dagger JV(0)\vec{s}_k. \tag{65}$$

The corresponding symplectic eigenvalues can be then easily inferred from Eq. (12).

On the other hand, for a generic choice of K, some matrix elements (61) diverge and some vanish exponentially with time. Thus, in this case, formally speaking there is no stationary solution to the equation considered. However, from a physical perspective, we focus on large rather than infinite times. From the point of view of the previous section, this may be interpreted as turning on the interaction with the environment for a given time, during which the system is subjected to a large but finite number of infinitesimal kicks. Note that, in general, such kicks are not energy preserving, since they may describe, e.g., squeezing transformations. In this regime, the covariance matrix becomes exponentially dominated by terms characterized in the first row of Eq. (63). An example of such dynamics is investigated by us in the next section.

VI. ENTANGLEMENT CREATION IN TWO-MODE STATES

To illustrate the results derived in the previous section, we consider an engineered dissipation scenario, in which we use the discussed evolution equation for creation of two-mode entanglement from a system initially in the vacuum state:

$$\hat{\rho}(0) = |00\rangle\langle 00|, \tag{66}$$

which is separable and Gaussian. For the evolution, we choose a single Lindblad operator from the one-parameter family of unitary two-mode squeezing operators:

$$\hat{L} = \hat{U}_r := e^{i\hat{h}_r}, \quad \hat{h}_r = -ir(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1\hat{a}_2), \tag{67}$$

where $r > 0$ is the squeezing strength and $\hat{a}_k := \frac{1}{2}(\hat{x}_k + i\hat{p}_k)$ is the annihilation operator for mode k.

Let us stress that, from the physical point of view, the evolution given by such a Lindblad operator is not at all equivalent to a "smooth" unitary evolution given by a squeezing Hamiltonian $\hat{H} = \hat{h}_r$. Instead, here, the squeezing should be understood as a series of regular, infinitely strong but infinitesimally short squeezing kicks, driving the system towards a high-energy state. In our case, the Hamiltonian behind the evolution is the kicked top Hamiltonian (46), with $\hat{h}_r$ entering at the level of the Dirac delta potential, as discussed in Sec. IV B.

We will proceed in two steps. First, we will certify that the evolved state is entangled. Then, we will quantify the amount of entanglement, showing that it is asymptotically unlimited.

A. Certifying entanglement

In the symplectic picture, the two-mode vacuum state is described by the covariance matrix

$$V(0) = \frac{1}{2}\mathbb{1}_4, \tag{68}$$

with $\vec{\xi}(0) = 0$. As for the squeezing operator, it is well known [65] that

$$
\begin{aligned}
\hat{U}_r^\dagger \hat{a}_1 \hat{U}_r &= \cosh r\,\hat{a}_1 + \sinh r\,\hat{a}_2^\dagger, \\
\hat{U}_r^\dagger \hat{a}_2 \hat{U}_r &= \sinh r\,\hat{a}_1^\dagger + \cosh r\,\hat{a}_2.
\end{aligned} \tag{69}
$$

Through Eqs. (28) and (29), we can see that the above transformation corresponds to the symplectic matrix

$$K_r = \begin{bmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{bmatrix}. \tag{70}$$

One can easily check that $K_r = \exp(JS_r)$ with

$$S_r = \begin{bmatrix} 0 & 0 & 0 & r \\ 0 & 0 & r & 0 \\ 0 & r & 0 & 0 \\ r & 0 & 0 & 0 \end{bmatrix}. \tag{71}$$

The matrix JS_r is normal and has the following eigendecomposition [the notation is the same as in Eq. (58)]:

$$
\begin{aligned}
s_1 &= -r, & \vec{s}_1 &= \frac{1}{\sqrt{2}}(0, 1, 0, 1)^T, \\
s_2 &= -r, & \vec{s}_2 &= \frac{1}{\sqrt{2}}(-1, 0, 1, 0)^T, \\
s_3 &= r, & \vec{s}_3 &= \frac{1}{\sqrt{2}}(0, -1, 0, 1)^T, \\
s_4 &= r, & \vec{s}_4 &= \frac{1}{\sqrt{2}}(1, 0, 1, 0)^T.
\end{aligned} \tag{72}
$$

Using the methodology developed in the previous section, we can easily calculate the matrix JV at any point in time. From the fact that $J^2 = -\mathbb{1}$, we then have $-J(JV) = V$, which explicitly reads

$$V(t) = \begin{bmatrix} A(t) & C(t) \\ C(t) & A(t) \end{bmatrix} \tag{73}$$

where

$$A(t) = \frac{1}{2} e^{2t \sinh^2 r} \cosh(t \sinh 2r) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C(t) = \frac{1}{2} e^{2t \sinh^2 r} \sinh(t \sinh 2r) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{74}$$

Having obtained the time-evolved covariance matrix, we can use it to certify that the corresponding state is entangled. In the symplectic picture, a sufficient condition for the presence of entanglement in the system is given by the positive partial transpose criterion for continuous variable systems [66,67]. The criterion states that, if the partial transposition of the state with respect to a given bipartition is not positive semidefinite, then the state is entangled with respect to this bipartition. In the case of two modes, the partially transposed state is not positive semidefinite, and thus the state is entangled, if [30]

$$\tilde{\nu}_- < 1/2, \tag{75}$$

where $\tilde{\nu}_-$ denotes the smallest symplectic eigenvalue of the covariance matrix of the partially transposed state:

$$V^{PT} = Q V Q, \quad Q = \text{diag}(1, 1, 1, -1). \tag{76}$$

Calculating the symplectic eigenvalues of V^{PT} via the eigenvalues of JV^{PT} as in Eq. (12), we find that, in the case at hand,

$$\tilde{\nu}_-(t) = \frac{1}{2} \exp[-(1 - e^{-2r})t]. \tag{77}$$

Evidently, the PPT criterion (75) for entanglement is fulfilled for all

$$t > 0. \tag{78}$$

In other words, despite being initially separable, the state of the system is entangled throughout the whole evolution.

B. Quantifying entanglement

We certified that the considered dissipative evolution drives the, initially separable, system into an entangled state. We will now proceed to assess how much entanglement is contained in the time-evolved state. To this end, we consider a measure of entanglement called *squashed entanglement*, one of the most prominent measures of entanglement [44,68,69]. For a generic bipartite state $\hat{\sigma}_{AB}$, squashed entanglement is defined as

$$\mathcal{E}_{\text{sq}}(\hat{\sigma}_{AB}) := \frac{1}{2} \inf_{\hat{\sigma}_{ABE}} I(A : B|E), \tag{79}$$

where $I(A : B|E) := S_V(\hat{\sigma}_{AE}) + S_V(\hat{\sigma}_{BE}) - S_V(\hat{\sigma}_E) - S_V(\hat{\sigma}_{ABE})$ is the conditional mutual information, $\hat{\sigma}_X$ are the (reduced) density operators of (sub)systems X, and the minimization is over all purifications $\hat{\sigma}_{ABE}$ of $\hat{\sigma}_{AB}$. Finally,

$$S_V(\hat{\sigma}) := -\text{Tr}\hat{\sigma} \ln \hat{\sigma} \tag{80}$$

is the von Neumann entropy.

Like other entanglement measures defined in terms of minimization over some set of states, squashed entanglement is notoriously difficult to calculate, being an NP-hard computation problem [70]. Here, we will not compute the squashed entanglement itself, but instead compute a lower bound for

it and show that it is an asymptotically unbounded function of time.

We begin by observing that, due to the extremality of Gaussian states with respect to continuous, superadditive entanglement measures [71], the squashed entanglement of any state $\hat{\sigma}$ is lower bounded by the squashed entanglement of a Gaussian state $\hat{\sigma}_G$ with the same covariance matrix. Furthermore, squashed entanglement of any state is lower bounded by so-called distillable entanglement $\mathcal{E}_{\text{dist}}$ [68], which, in turn, is lower bounded by the coherent information [72,73]

$$I_{\mathcal{C}}(\hat{\sigma}) := S_V(\hat{\sigma}_A) - S_V(\hat{\sigma}), \tag{81}$$

where $\hat{\sigma}_A = \text{Tr}_B \hat{\sigma}$.

In our case, this means that we have the following chain of inequalities:

$$\mathcal{E}_{\text{sq}}[\hat{\rho}(t)] \geqslant \mathcal{E}_{\text{sq}}[\hat{\rho}_G(t)] \geqslant \mathcal{E}_{\text{dist}}[\hat{\rho}_G(t)]$$
$$\geqslant I_{\mathcal{C}}[\hat{\rho}_G(t)] = S_V[\hat{\rho}_{G,A}(t)] - S_V[\hat{\rho}_G(t)], \tag{82}$$

where $\hat{\rho}_G(t)$ is a Gaussian state with the same covariance matrix (73) as our state and $\hat{\rho}_{G,A}(t) = \text{Tr}_B \hat{\rho}_G(t)$. Crucially, both von Neumann entropies on the right-hand side are simple functions of the symplectic eigenvalues of the respective state. Let us define the auxiliary function:

$$f(x) := (x + 1/2) \ln(x + 1/2) - (x - 1/2) \ln(x - 1/2). \tag{83}$$

Then, for a one- or two-mode Gaussian state $\hat{\sigma}_G$ with covariance matrix $V_{\hat{\sigma}}$ [74]

$$S_V(\hat{\sigma}_G) = \sum_{j=1}^N f[\nu_j(V_{\hat{\sigma}})], \tag{84}$$

where N is the number of modes. In the case at hand, it is easy to calculate that the symplectic eigenvalues of the covariance matrix (73) equal

$$\nu_1(t) = \nu_2(t) = \frac{1}{2} e^{2t \sinh^2 r} \equiv \nu(t). \tag{85}$$

On the other hand, one can easily see from the definition (4) that the reduced covariance matrix V_A corresponding to the first mode is given by the upper-left block of (73), i.e., $V_A = A(t)$. The only symplectic eigenvalue of V_A equals

$$\nu_A(t) = \frac{1}{2} \cosh(t \sinh 2r) e^{2t \sinh^2 r}. \tag{86}$$

Using the last four equations in Eq. (82), we finally obtain

$$\mathcal{E}_{sq}[\hat{\rho}(t)] \geqslant I_{\mathcal{C}}[\hat{\rho}_G(t)] = f[\nu_A(t)] - 2f[\nu(t)]. \tag{87}$$

The above lower bound for squashed entanglement, and therefore also squashed entanglement itself, grows indefinitely. To show this, we first calculate that

$$I_{\mathcal{C}}[\hat{\rho}_G(t)] = e^{2z} \ln \tanh z + \ln \frac{2\{e^{2z} \cosh[2 \coth(r)z] + 1\}}{e^{4z} - 1}, \tag{88}$$

where we denoted $z := t \sinh^2 r$ for shortness. For very large t, corresponding to very large z, the first term on the right-hand side approaches the constant value of -2. In the second term, $\cosh[2 \coth(r)z]$ approaches $e^{2 \coth(r)z}/2$, which means

that the logarithm behaves like $\ln e^{4z[\coth(r)-1]} = 4z[\coth(r) - 1]$. It follows that

$$
\begin{aligned}
I_C[\hat{\rho}_G(t)] \xrightarrow[t\to\infty]{} & -2 + 4z[\coth(r) - 1] \\
& = -2 + 4t \sinh^2(r)[\coth(r) - 1],
\end{aligned} \tag{89}
$$

where, to be explicit, in the bottom line we went back to the parametrization in terms of t. Clearly, this is a linear function in t with positive slope, since $\coth(r) > 1$ for all $r > 0$. Therefore, $I_C[\hat{\rho}_G(t)]$ is asymptotically infinite, and thus the same is also true for squashed entanglement itself. This is what we wanted to show.

VII. CONCLUDING REMARKS

Motivated by recent findings in resource theories of non-Gaussianity, we developed a model of dissipative evolution which preserves the set of quantum Gaussian states without preserving the set of Gaussian states itself. We showed that, while such a model constitutes a natural description of random scattering, it can also be applied to engineered dissipation, as showcased through an example of entanglement creation in two-mode states. Finally, the model is fully compatible with the symplectic (covariance matrix) picture of quantum states, allowing one to study it with the same tools that are typically used for Gaussian states.

Besides applications to phenomena that include random scattering, as well as engineered dissipation, our findings suggest the following directions for future research. To start with, let us briefly denote the generator of Gaussian evolution (19) by $\mathcal{L}_G$ and the generator of the evolution (22) based on unitary Lindbladians by $\mathcal{L}_{cG}$. Because both $\mathcal{L}_G$ and $\mathcal{L}_{cG}$ preserve the set of quantum Gaussian states, then, by Trotter's formula [75]

$$
e^{(\mathcal{L}_G + \mathcal{L}_{cG})t} = \lim_{n\to\infty} (e^{\mathcal{L}_G t/n} e^{\mathcal{L}_{cG} t/n})^n \tag{90}
$$

the combined generator $\mathcal{L}_G + \mathcal{L}_{cG}$ also does. Therefore, from the point of view of dynamics of quantum Gaussian states, the discussed generator can be seen not only as an alternative to the Gaussian model, but also as its extension. For example, it could be used to introduce generic quantum Gaussian noise, especially in the form of the scattering integral (48), into an otherwise Gaussian system.

Furthermore, while operations preserving the set of Gaussian states are fully characterized [76], an analogous problem was not resolved for quantum Gaussian states, partially due to the lack of one-to-one correspondence with the set of states with positive Wigner distribution [77]. This leads to the following question: what other evolution models preserve the set of quantum Gaussian states but not the set of Gaussian states? What physical scenarios can they describe? An immediate generalization of our results would be to replace $\mathcal{L}_{cG}$ by $\mathcal{L} \cdot = \sum_k (\theta_k \cdot -\hat{\mathbb{1}})$, with θ_k being arbitrary Gaussian channels. One can easily check that such generator preserves the set of quantum Gaussian states. It would be interesting to see if this is the most general generator with this property, and if not, how it could be generalized further.

APPENDIX: REWRITING THE OPERATOR (42) IN TERMS OF A KICKED TOP HAMILTONIAN

In this Appendix, we show that the unitary operator (42) can be obtained by substituting the kicked top Hamiltonian $\hat{w}_n = \hat{H}_{\mathrm{KT}}$ with inputs (47) into Eq. (40). In other words, we show that the operators

$$
\hat{X}_n = \left(\hat{\mathbb{1}} \otimes |0\rangle\langle 0| + \sum_{j=1}^{M} \hat{U}_j \otimes |j\rangle\langle j| \right) \hat{O}(\Delta t), \tag{A1}
$$

$$
\hat{Y}_n = \mathcal{T} \exp\left(-i \int_{(n-1)\Delta t}^{n\Delta t} d\tau \, \hat{w}_n(\tau) \right) \tag{A2}
$$

are identical for

$$
\hat{w}_n(\tau) = \hat{o}_n(\tau) + \delta(\tau - n\Delta t) \sum_{j=1}^{M} \hat{h}_j \otimes |j\rangle\langle j|, \tag{A3}
$$

with $\hat{U}_j = e^{-i\hat{h}_j}$ and

$$
\hat{O}(\Delta t) = \mathcal{T} \exp\left(-i \int_{(n-1)\Delta t}^{n\Delta t} d\tau \, \hat{o}_n(\tau) \right). \tag{A4}
$$

We begin by observing that $\hat{Y}_n$ can be recast into

$$
\begin{aligned}
\hat{Y}_n = \lim_{\epsilon\to 0} & \, \mathcal{T} \exp\left(-i \int_{n\Delta t-\epsilon/2}^{n\Delta t+\epsilon/2} d\tau \, \hat{w}_n(\tau) \right) \\
& \times \mathcal{T} \exp\left(-i \int_{(n-1)\Delta t}^{n\Delta t-\epsilon/2} d\tau \, \hat{w}_n(\tau) \right).
\end{aligned} \tag{A5}
$$

Provided $\hat{o}_n$ is a well-behaved function of time [which can be inferred from the well-behaved nature of its exponential (43)], its contribution to the first integral vanishes in the limit. At the same time, the delta distribution integrates to one. See, e.g., [78] for rigorous treatment. In conclusion,

$$
\mathcal{T} \exp\left(-i \int_{n\Delta t-\epsilon/2}^{n\Delta t+\epsilon/2} d\tau \, \hat{w}_n(\tau) \right) \to e^{-i\sum_{j=1}^{M} \hat{h}_j \otimes |j\rangle\langle j|}. \tag{A6}
$$

In the second integral, the situation is reversed. Because the integral does not contain the point $\tau = n\Delta t$, the delta distribution does not contribute and we can simply put $\hat{w}_n = \hat{o}_n$. Thus, based on Eq. (A4),

$$
\mathcal{T} \exp\left(-i \int_{(n-1)\Delta t}^{n\Delta t-\epsilon/2} d\tau \, \hat{w}_n(\tau) \right) \to \hat{O}(\Delta t). \tag{A7}
$$

Combining the last three expressions, we obtain

$$\hat{Y}_n = e^{-i \sum_{j=1}^{M} \hat{h}_j \otimes |j\rangle\langle j|} \hat{O}(\Delta t). \tag{A8}$$

Because the generator of the exponential on the right-hand side is diagonal in the second subsystem's number basis, the

exponentiation can be explicitly performed, quickly yielding

$$e^{-i \sum_{j=1}^{M} \hat{h}_j \otimes |j\rangle\langle j|} = \hat{\mathbb{1}} \otimes |0\rangle\langle 0| + \sum_{j=1}^{M} \hat{U}_j \otimes |j\rangle\langle j|. \tag{A9}$$

Clearly, this makes Eq. (A8) identical to (A1), which is what we wanted to prove.

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Reduced state of the field and classicality of quantum Gaussian evolution

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The notion of classicality of quantum evolution of light is an object of both conceptual and practical importance. The main goal of this work is to derive the exact conditions for the classicality of quantum Gaussian evolution, i.e., the evolution of Gaussian states of light and their convex combinations, a model which is of great significance in quantum optics and information. According to our findings, quantum Gaussian evolution should be considered classical if the Hamiltonian and Lindblad operators generating it correspond to passive optical transformations. This is illustrated with several explicit examples, ranging from Gaussian thermal operations to entanglement-maximizing dissipative engineering. Our results are obtained using the recently introduced mesoscopic formalism of the reduced state of the field, which was originally devised as as a description of macroscopic quantum fields. Here, to make the framework suitable for our goal, we redevelop it as a tool for probing classicality, which constitutes our second main contribution.

I. INTRODUCTION

Classicality of light has been a subject of an ongoing debate at least since Einstein's work on photoelectric effect [1] and the discovery of wave-particle duality [2]. While it is generally believed that, e.g., Glauber's coherent states [3] are more classical than pure Fock states with the same mean particle number [4], there exists no widely accepted criterion for classicality of multiphoton states of light. Even classicality of a single photon continues to be vividly discussed [5,6].

Similar considerations concern the time evolution of the electromagnetic field. Quantum particles evolve under the von Neumann equation, while classical particles evolve under Liouville's equation. The degree to which the latter approximates the former is quantified by the relation between the energy scales in the system and the Planck's constant. However, the evolution of both the classical and quantum electromagnetic fields is given by the same set of four Maxwell's equations [7,8].

In modern quantum optics, the electromagnetic field is typically described by second quantization, with the occupation numbers of photons of a given frequency described by the density operator in an infinitely-dimensional Hilbert space [8,9]. Currently available experimental operations, which describe the time evolution of this density operator, consist primarily of Gaussian operations and measurements [10,11], prominent in, e.g., quantum key distribution and other information processing tasks [12–14].

To this day, the classicality of Gaussian evolution and, more broadly, quantum evolution of light and Gaussian wave packets, was investigated using a number of methods: Relying on phase-space and the Winger distribution, hybridization of

quantum and classical theories, and path integrals [15–18]. Besides its conceptual significance, identifying classicality of evolution of light is important in practice, since classical description is typically much simpler than quantum theory [19–21].

In this article, we approach the problem from the point of view of the reduced state of the field (RSF) [22], a recent mesoscopic theory [23] of many-particle bosonic systems. Relying solely on the first two moments of the mode creation and annihilation operators, the description reduces the infinitely-dimensional density operator of the *N* mode field to an *N*-dimensional matrix defining the aforementioned RSF. Originally, RSF was designed to describe the quantum features of macroscopic fields of a single particle type, including light fields. In particular, the formalism was successfully applied to polarization optics, bridging the Mueller and Jones calculi, as well as to shock wave generation [22].

Here, we employ RSF to isolate the classical subclass of quantum Gaussian evolution, defined as evolution preserving the set of Gaussian states and their convex combinations. We do this in two steps. First, we investigate the formalism itself to show that, complementarily to its original goal, RSF also captures the classical aspects of quantum fields. In particular, we prove that RSF contains limited information about bipartite entanglement, if any, and that the entropic description in terms of RSF closely resembles that given by the semiclassical Wehrl entropy [24,25]. In this way, we establish RSF as a tool for studies of classicality within quantum mechanics.

Second, we compare quantum Gaussian evolution with the time evolution model built into the RSF framework, deriving in this fashion the explicit subset of Gaussian evolution which is classical with respect to the RSF toolbox. The classicality of the obtained evolution is intuitive, as it consists exclusively of passive transformations, which correspond to experimental operations that can be successfully understood

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by treating light as a classical wave, such as beam-splitters and phase-shifters. On the contrary, evolution employing quantum squeezing does not fit in the RSF framework.

This work is organized as follows. In Sec. II, we briefly summarize the main subject of our work–quantum Gaussian evolution. Over the course of the next three sections, we introduce the formalism of RSF and investigate its various aspects with respect to their classicality: Correlations in Sec. III, entropy in Sec. IV, and time evolution in Sec. V. Finally, in Sec. VI, we use the RSF formalism to derive the classical subset of the quantum Gaussian evolution family. We conclude in Sec. VII.

Notational remark

In this work, we employ three different formalisms: The standard, density operator picture, the symplectic picture and the RSF framework. For clarity, we use different notation for operators in each of these pictures. Operators associated with the the standard picture are denoted by hats, e.g., $\hat{\rho}$. Operators associated with the symplectic picture are denoted by capital letters with no hats, e.g., V. Operators associated with the RSF framework are denoted by small letters, also with no hats, e.g., r.

II. QUANTUM GAUSSIAN EVOLUTION OF LIGHT

We begin by introducing the main subject of our considerations: Quantum Gaussian evolution. To this end, we also briefly summarize the notions of Gaussian states and symplectic picture, which will serve as important tools in the derivation of our findings.

A. Gaussian states and symplectic picture

We consider an N-mode Hilbert space spanned by the set of N pairs of mode quadratures collected in the vector

$$\hat{\vec{\xi}} := (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_N, \hat{p}_N)^T, \tag{1}$$

where $\hat{x}_k$ and $\hat{p}_k$ fulfill the canonical commutation relations:

$$[\hat{x}_k, \hat{p}_{k'}] = i\delta_{kk'}, \quad [\hat{x}_k, \hat{x}_{k'}] = [\hat{p}_k, \hat{p}_{k'}] = 0, \tag{2}$$

where we set $\hbar = 1$. Since the mode quadratures form a basis of operators acting on the N-mode Hilbert space, the state of the system is fully described by the complete collection of correlation functions of the form

$$\langle \hat{\xi}_{l_1} \ldots \hat{\xi}_{l_n} \rangle := \text{Tr}[\hat{\rho}\, \hat{\xi}_{l_1} \ldots \hat{\xi}_{l_n}]. \tag{3}$$

In the case of Gaussian states, defined as states with normal (Gaussian) characteristic functions and quasiprobability distributions [8,26,27], the complete information about the system is contained within only the one- and two-point correlation functions, i.e., with $n = 1, 2$ in the equation above. The former are contained in the vector of means

$$|\xi\rangle := \sum_{k=1}^{2N} \langle \hat{\xi}_k \rangle |k\rangle, \tag{4}$$

while the latter are encoded in the matrix of second moments

$$V := \frac{1}{2} \sum_{k,k'=1}^{2N} \langle \{\hat{\xi}_k, \hat{\xi}_{k'}\} \rangle |k\rangle \langle k'|. \tag{5}$$

Often, instead of V, one uses the covariance matrix, defined as $V_{\text{cov}} = V - |\xi\rangle\langle\xi|$.

Any valid covariance matrix has to fulfill the Heisenberg uncertainty principle:

$$\sqrt{\langle \hat{x}_k^2 \rangle - \langle \hat{x}_k \rangle^2} \sqrt{\langle \hat{p}_k^2 \rangle - \langle \hat{p}_k \rangle^2} \geqslant \frac{1}{2}, \tag{6}$$

where $k \in \{1, \ldots, N\}$, equivalent to [26]

$$V_{\text{cov}} - \frac{i}{2}J \geqslant 0. \tag{7}$$

Here, J is the *symplectic form*, defined as

$$J := -i \sum_{k,k'=1}^{2N} [\hat{\xi}_k, \hat{\xi}_{k'}]|k\rangle\langle k'|, \tag{8}$$

and explicitly equal to

$$J = \bigoplus_{k=1}^{N} J_2, \quad J_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{9}$$

The symplectic form defines the symplectic group $Sp(2N, \mathbb{R})$ consisting of matrices S of size $2N \times 2N$, such that $SJS^T = J$.

As a matter of fact, the pair $(V, |\xi\rangle)$ contains the same information as $(V_{\text{cov}}, |\xi\rangle)$, and both in the same way define the *symplectic picture* of quantum states (sometimes referred to as covariance matrix picture), which is a convenient description of the first two moments of the system, particularly in the case of Gaussian states and dynamics. Here, we employ the pair $(V, |\xi\rangle)$, since, as we will see in the next section, it is by construction closer to the reduced state of the field than $(V_{\text{cov}}, |\xi\rangle)$.

B. Quantum Gaussian time evolution

The time evolution of quantum open systems is typically modeled by the *Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) equation* (also known as the Lindblad equation) [28–30], which in the diagonalized form reads:

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{L}_k^\dagger \hat{L}_k, \hat{\rho}\} \right), \tag{10}$$

where $\hat{H}$ denotes the system Hamiltonian and $\hat{L}_k$ are the Lindblad operators.

One of the main sources of motivation for studying Gaussian states is that, due to technical limitations, in practice we are often restricted to Hamiltonians that are polynomials of at most second degree in mode quadratures:

$$\hat{H} = \frac{1}{2}\hat{\vec{\xi}}^T G \hat{\vec{\xi}}, \tag{11}$$

where G is a $2N \times 2N$, real, symmetric matrix. The structure-preserving evolution of Gaussian states is driven by precisely such Hamiltonians.

Similarly, to preserve Gaussianity of an initial state along the course of time evolution, the Lindblad operators need to

be linear in mode quadratures [31]:

$$\hat{L}_k = \vec{c}_k \cdot \hat{\vec{\xi}}, \quad \vec{c}_k \in \mathbb{C}^{2N}, \tag{12}$$

so that the resulting dissipator is a polynomial of second degree in mode quadratures, like the Hamiltonian.

However, the same experimental tools that let one create and manipulate Gaussian states can be used to create and manipulate convex combinations of Gaussian states. In fact, according to recent theories of non-Gaussianity [10,11], from the point of view of useful non-Gaussianity, there is no difference between Gaussian states and their convex combinations. Only states that cannot be written as a convex combination of Gaussian states are genuinely non-Gaussian, or *quantum non-Gaussian* [32–34]. Consequently, states that are either Gaussian or can be written as a convex combination of Gaussian states are called *quantum Gaussian*.

For this reason, in addition to linear Lindblad operators, which preserve the set of Gaussian states, we also consider unitary Lindblad operators

$$\hat{L}_k = \sqrt{\gamma_k}\hat{U}_k, \tag{13}$$

where $\gamma_k \geq 0$, $\sum_k \gamma_k = 1$ and $\hat{U}_k$ are unitary operators such that $\hat{U}_k = \exp(i\hat{g}_k)$ with $\hat{g}_k$ being polynomials of at most second degree in the mode quadratures. The dynamics induced by such Lindblad operators does not preserve the set of Gaussian states, but preserves the set of quantum Gaussian states [35]. Dissipators of this form are most well-known for describing random scattering, see Refs. [36–38].

Written in the symplectic picture, the evolution given by the Hamiltonian (11) and Lindblad operators (12), (13) reads [39–41]

$$\frac{d}{dt}V = F_G(V) + F_L(V) + F_U(V),$$
$$\frac{d}{dt}|\xi\rangle = f_G(|\xi\rangle) + f_L(|\xi\rangle) + f_U(|\xi\rangle). \tag{14}$$

Here,

$$F_G(V) := JGV - VGJ,$$
$$f_G(|\xi\rangle) := JG|\xi\rangle, \tag{15}$$

are the Hamiltonian terms, while [42]

$$F_L(V) := JI_CV + VI_CJ + JR_CJ^T,$$
$$f_L(|\xi\rangle) := JI_C|\xi\rangle, \tag{16}$$

with $R_C \equiv \mathrm{re}C^\dagger C$, $I_C \equiv \mathrm{im}C^\dagger C$ and $C_{kl} := (\vec{c}_k)_l$ stem from linear Lindblad operators (12). We remark that the Gaussianity-preserving time evolution given by these functions is known to have exact solutions [43–45] and is well-studied using Green functions [46,47] and, in particular, the symplectic picture [39–41].

The final terms

$$F_U(V) := \sum_j \gamma_j\big(K_j V K_j^T - V\big),$$
$$f_U(|\xi\rangle) := \sum_j \gamma_j(K_j|\xi\rangle - |\xi\rangle), \tag{17}$$

where K_j are symplectic, stem from the unitary Lindblad operators (13) and represent a relatively novel type of dynamics that preserves only the set of quantum Gaussian states [35].

Equation (14) defines the *quantum Gaussian evolution*. The ultimate goal of our article is to identify the subclass of semiclassical evolution consistent with this equation. Before we can do that, however, we need to develop the necessary tools to achieve this goal, namely, the framework of the reduced state of the field (RSF).

III. REDUCED STATE OF THE FIELD AS A CLASSICAL DESCRIPTION OF BOSONIC FIELDS

In this section, we summarize the relevant information about RSF and simultaneously investigate it to show that it provides a semiclassical description for bosonic many-particle fields, thus constituting a viable tool for our main goal.

A. Reduced state of the field (RSF)

The main idea behind RSF was to describe many-particle, or macroscopic, quantum fields. In such a case, instead of using the mode quadratures, it is often more convenient to use the annihilation and creation operators

$$\hat{a}_k := \frac{1}{\sqrt{2}}(\hat{x}_k + i\hat{p}_k), \quad \hat{a}_k^\dagger = \frac{1}{\sqrt{2}}(\hat{x}_k - i\hat{p}_k), \tag{18}$$

with the canonical commutation relations (2) now reading

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0. \tag{19}$$

An arbitrary *n*-particle state in the many-body Hilbert space can be then constructed by acting on the vacuum state with *n* appropriate creation operators.

In the case of macroscopic fields, typically modeled as noninteracting fields with dynamics governed by field equations linear in creation and annihilation operators with possible external coherent sources, the fundamental observables are either *additive*, like energy [22],

$$\hat{O} = \sum_{k,k'=1}^N o_{kk'}\hat{a}_k^\dagger\hat{a}_{k'}, \tag{20}$$

or *linear*, like momentum,

$$\hat{\sigma} = \sum_{k=1}^N (\sigma_k^*\hat{a}_k + \sigma_k\hat{a}_k^\dagger). \tag{21}$$

One can easily check that the expectation values of such observables can be equivalently rewritten as

$$\mathrm{Tr}\hat{\rho}\,\hat{O} = \mathrm{tr}ro, \quad \mathrm{Tr}\hat{\rho}\,\hat{\sigma} = \langle\sigma|\alpha\rangle + \langle\alpha|\sigma\rangle, \tag{22}$$

where

$$r := \sum_{k,k'=1}^N \mathrm{Tr}[\hat{\rho}\,\hat{a}_{k'}^\dagger\hat{a}_k]|k\rangle\langle k'| \tag{23}$$

defines the *single-particle density matrix*,

$$|\alpha\rangle := \sum_{k=1}^N \mathrm{Tr}[\hat{\rho}\,\hat{a}_k]|k\rangle \tag{24}$$

defines the *averaged field*, while

$$o = \sum_{k,k'=1}^{N} o_{kk'} |k\rangle\langle k'|, \quad |\sigma\rangle = \sum_{k=1}^{N} \sigma_k |k\rangle \quad (25)$$

are the *reduced observables* corresponding to $\hat{O}$ and $\hat{\sigma}$.

The single-particle density matrix contains information about mode occupation and coherence in the state. In particular, its diagonal elements equal the mean particle numbers: $r_{kk} = \langle \hat{a}_k^\dagger \hat{a}_k \rangle = \langle \hat{n}_k \rangle$ and, consequently, the matrix is normalized to the mean total particle number: $\text{tr} r = \sum_{k=1}^{N} \langle \hat{n}_k \rangle \equiv \langle \hat{n} \rangle$. Note that, by construction, the single-particle density matrix is nonnegative.

The averaged field contains information about local phases of the field. For example, in the case of a pure Fock state, the phase is undefined, yielding no such information, with the opposite situation in the case of a pure coherent state.

Together, the single particle density matrix and the averaged field constitute the *reduced state of the field (RSF)* associated with the density operator $\hat{\rho}$ [22]. A major advantage of the RSF formalism is that it is, to a large degree, self-contained, in the sense that it allows for study of a variety of phenomena without having to refer to other frameworks. In particular, it comes equipped with its own definition of entropy and a time evolution model, both of which are investigated by us in the subsequent sections.

In the case of additive and linear observables (20), (21), the RSF description is complete. In the case where the observables of interest are more general, RSF describes a subset of degrees of freedom of the system. We now proceed to give a physical interpretation for the degrees of freedom contained within RSF.

B. Physical meaning of correlations within the RSF framework

To see what physical information is associated with the degrees of freedom contained within RSF, we begin by observing that RSF is related to the symplectic picture of quantum states via

$$r = \mathcal{R} V \mathcal{R}^\dagger - \frac{1}{2} \mathbf{1}_N, \quad |\alpha\rangle = \mathcal{R} |\xi\rangle, \quad (26)$$

where we use $\mathbf{1}_N$ to denote the identity matrix of size $N \times N$, and

$$\mathcal{R} := \frac{1}{\sqrt{2}} \sum_{k=1}^{N} |k\rangle [\langle 2k-1| + i\langle 2k|] \quad (27)$$

defines the *reduction matrix*. The Heisenberg uncertainty principle (7) translates to nonnegativity of the *correlation matrix*:

$$r_\alpha := r - |\alpha\rangle\langle\alpha| \geqslant 0, \quad (28)$$

which was defined already in Ref. [22]. Note that it follows immediately from Eq. (26) that

$$r_\alpha = \mathcal{R}(V - |\xi\rangle\langle\xi|)\mathcal{R}^\dagger - \frac{1}{2}\mathbf{1}_N = \mathcal{R} V_{\text{cov}} \mathcal{R}^\dagger - \frac{1}{2}\mathbf{1}_N. \quad (29)$$

The relations (26) and (28) are derived by us in Appendix A.

The input of the reduction matrix belongs to a $2N$-dimensional space, while the output is only N-dimensional. Clearly, then, the reduction matrix cuts some of the information from the symplectic picture. As we will now show,

this missing information is relevant for practical scenarios requiring bipartite quantum entanglement. For a given entangled state to be useful for any such task, e.g., quantum code encryption or teleportation, it first needs to be *distilled* [48].

Crucially, not every entangled state is distillable. A necessary condition for bipartite entanglement distillation is given by the *positive partial transpose (PPT) criterion* [49,50], originally stated as a necessary condition for separability. Adopted to the language of distillable entanglement, the PPT criterion states that if the partial transposition of the state with respect to a given bipartition is positive semidefinite, then the state does not contain distillable entanglement with respect to this bipartition [48].

In the symplectic picture, partial transposition of arbitrary chosen modes is performed by replacing the covariance matrix by

$$V_{\text{cov},\tilde{q}} = Q_{\tilde{q}} V_{\text{cov}} Q_{\tilde{q}}, \quad (30)$$

with

$$Q_{\tilde{q}} = \text{diag}(1, q_1, \ldots, 1, q_N), \quad (31)$$

where $q_k = -1$ for modes that are being transposed and $q_k = 1$ otherwise.

From the perspective of distillable entanglement, the PPT criterion for continuous variable systems states that if [cf. Eq. (7)]

$$V_{\text{cov},\tilde{q}} - \frac{i}{2} J \geqslant 0, \quad (32)$$

then the state does not contain distillable entanglement with respect to the bipartion given by $\tilde{q}$ [48,50]. Therefore, violation of Eq. (32) indicates its presence. Note that if this inequality holds, then the state may still contain so-called *bound* entanglement. This type of entanglement is, however, much less useful in practice.

We will now show that in the RSF picture, the PPT criterion is undecidable. In turn, the formalism contains no information about distillable entanglement. To this end, it is enough to limit our considerations to the correlation matrix (28), since the averaged field contains only local information and is therefore irrelevant for entanglement.

The key observation is that among all the covariance matrices corresponding to a given correlation matrix r_α through Eq. (29), there is one that equals

$$\bar{V}_{\text{cov}} = r_\alpha \otimes \mathbf{1}_2 + \frac{1}{2} \mathbf{1}_{2N}. \quad (33)$$

This can be seen as follows. First, as is easy to compute, $\mathcal{R}(r_\alpha \otimes \mathbf{1}_2)\mathcal{R}^\dagger = r_\alpha$, from which it immediately follows that the correlation matrix corresponding to $\bar{V}_{\text{cov}}$ is indeed equal to r_α. Second, $\bar{V}_{\text{cov}}$ fulfills the Heisenberg uncertainty principle (7), since

$$\bar{V}_{\text{cov}} - \frac{i}{2} J = r_\alpha \otimes \mathbf{1}_2 + \frac{1}{2}(\mathbf{1}_{2N} - iJ). \quad (34)$$

By construction, r_α is nonnegative, and thus so is the first term on the right-hand side (r.h.s.). The second term, however, can be decomposed into N blocks of size 2×2 as $\mathbf{1}_{2N} - iJ = \bigoplus_{j=1}^{N}(\mathbf{1}_2 - iJ_2)$. It is straightforward to calculate that each of these blocks is nonnegative, making the whole matrix

nonnegative. Therefore, Eq. (34) is nonnegative and $\bar{V}_{\text{cov}}$ is a valid covariance matrix.

Crucially, all variants of partial transposition of $\bar{V}_{\text{cov}}$ satisfy the condition (32). Indeed, for any $\vec{q}$ we have

$$\bar{V}_{\text{cov},\vec{q}} - \frac{i}{2}J = Q_{\vec{q}}(r_\alpha \otimes \mathbf{1}_2)Q_{\vec{q}} + \frac{1}{2}(\mathbf{1}_{2N} - iJ), \qquad (35)$$

where we used the fact that $Q_{\vec{q}}^2 = \mathbf{1}_{2N}$. As before, the second term is nonnegative. However, the same is also true for the first term since, due to unitarity and Hermiticity of $Q_{\vec{q}}$, the eigenvalues of $Q_{\vec{q}}(r_\alpha \otimes \mathbf{1}_2)Q_{\vec{q}}$ are the same as the eigenvalues of $r_\alpha \otimes \mathbf{1}_2$.

Consequently, every RSF description corresponds to at least one symplectic description of a system that fulfills the PPT criterion, making this criterion trivial from the point of view of the RSF framework. This leads to the following proposition.

Proposition 1. The RSF framework contains no information about bipartite distillable entanglement.

This proposition has special consequences for two-mode Gaussian states, for which the PPT criterion is equivalent to the presence of any form of entanglement, not only distillable entanglement [48]:

Corollary 2. In the case of two-mode Gaussian states, the RSF framework contains no information about any form of entanglement.

We conjecture that these findings hold in general, i.e., RSF contains no information about any type of quantum entanglement in any quantum state. Irrespectively, Proposition 1 and Corollary 2 show that the ability to describe entanglement within the RSF formalism is severely limited, strongly suggesting the framework to be semiclassical.

IV. CLASSICALITY OF RSF ENTROPY

The fact that RSF contains limited information about entanglement strongly suggests it is a semiclassical formalism. To further reinforce this interpretation, in this section, we analyze the entropic description in terms of RSF, showing that it is similar to the one given by the semiclassical Wehrl entropy.

A. Reduced entropy

The standard choice for quantum (information) entropy is given by the von Neumann entropy [51]

$$S_V(\hat{\rho}) := -\text{Tr}\,\hat{\rho} \ln \hat{\rho}, \qquad (36)$$

where we set $k_B = 1$. Because of its information-theoretic origin as a generalization of the Shannon entropy, the von Neumann entropy is most easily interpreted as a measure of uncertainty about the state of the system. The von Neumann entropy is invariant under all unitary transformations and it attains its minimum value—zero—for all pure states.

To describe entropy within the RSF formalism, one needs to find a way to derive a valid entropy measure that depends only on the components of RSF. In Ref. [22], this was done with the use of the *maximum entropy principle* [52,53]. According to this principle, given only a partial knowledge about a physical system, one should assume the highest

possible value of entropy consistent with this knowledge. Interpreting entropy as the amount of uncertainty about the state of the system, the maximum entropy principle means simply that one should not presume to be more certain about the system's state than their knowledge lets them.

For example, if one has absolutely no knowledge about which quantum state the system is in, one should assume it to be maximally mixed, i.e., $\hat{\rho} = \hat{\mathbf{1}}_\Omega/\Omega$, where Ω denotes the number of possible orthogonal system states, or equivalently the dimension of the Hilbert space. This is because such density operator is the only one for which all system states are equally probable.

In this case, it is easy to calculate that the von Neumann entropy coincides with the classical Boltzmann entropy

$$S_B = S_V(\hat{\mathbf{1}}_\Omega/\Omega) = \ln\Omega. \qquad (37)$$

Viewed from the perspective of the maximum entropy principle, the Boltzmann entropy is simply the maximum value of the von Neumann entropy consistent with having no knowledge about the quantum state of the system.

In the RSF formalism, the only information we have about the system is its RSF. Thus, according to the maximum entropy principle, we should assume that the system's von Neumann entropy has the highest value possible for a system with that specific RSF. As was calculated in Ref. [22], among all the quantum states with the same RSF $(r, |\alpha\rangle)$, the von Neumann entropy is maximal for the thermal-like state

$$\hat{\rho}_{r,|\alpha\rangle} = \frac{1}{z}D(\vec{\alpha})\exp\left(-\sum_{k,k'=1}^{N} r_{kk'}\hat{a}_k^\dagger \hat{a}_k\right)D^\dagger(\vec{\alpha}), \qquad (38)$$

where

$$z = \text{Tr}\exp\left(-\sum_{k,k'=1}^{N} r_{kk'}\hat{a}_k^\dagger \hat{a}_k\right) \qquad (39)$$

and

$$D(\vec{\alpha}) = \exp\left[\sum_{k=1}^{N}(\alpha_k^* \hat{a}_k - \alpha_k \hat{a}_k^\dagger)\right] \qquad (40)$$

is the (unitary) N-mode displacement operator.

Thus, the RSF entropy can be defined as [22]

$$s_v(\hat{\rho}) := S_V(\hat{\rho}_{r,|\alpha\rangle})$$
$$= \text{tr}[(r_\alpha + \mathbf{1}_N)\ln(r_\alpha + \mathbf{1}_N) - r_\alpha \ln r_\alpha], \qquad (41)$$

with the correlation matrix r_α as in Eq. (28). In accordance with the maximum entropy principle, such entropy, dubbed *reduced entropy* [22], is simply the maximum value of the von Neumann entropy consistent with having no knowledge about the quantum state of the system except for its RSF.

The reduced entropy satisfies the natural condition $s_v(\hat{\rho}) \geqslant 0$, with equality if and only if the correlation matrix is equal to zero, which happens only when the density operator of the field is given by a coherent state. In contrast, the von Neumann entropy vanishes for any pure state.

While based on sound principles, the reduced entropy lacks a clear physical interpretation. We now proceed to investigate the qualitative and quantitative features of this entropy,

showing that is has a semiclassical character akin to the Wehrl entropy.

B. Reduced Wehrl entropy and its classical features

The Wehrl entropy [24] is defined as the continuous Shannon entropy of the Husimi Q representation of the quantum state:

$$S_W(\hat{\rho}) := - \int \frac{d^{2N}\vec{\beta}}{\pi^N} Q(\vec{\beta}) \ln Q(\vec{\beta}). \tag{42}$$

Here, $Q(\vec{\beta}) = \langle \vec{\beta} | \hat{\rho} | \vec{\beta} \rangle$ is the Husimi Q representation [54] of the state $\hat{\rho}$, $|\vec{\beta}\rangle$ is an N-mode coherent state and the integration is over the real and imaginary parts of every component of the complex vector $\vec{\beta}$.

The Wehrl entropy is typically considered to be a semiclassical approximation to the von Neumann entropy, since it is constructed by replacing the quantum density operator in the definition of the von Neumann entropy by its representation $Q(\vec{\beta})$ in the phase-space [25,55]. The two entropies differ significantly. Unlike the von Neumann entropy, the Wehrl entropy attains its minimum value, N, only for coherent states [56]. Furthermore, it is not invariant under all unitary transformations of the state.

Looking at the reduced entropy (41), we can see that it possesses the same qualities. The fact that it is minimized by coherent states was already discussed. As for invariance under unitary operations, consider, e.g., the transformation $\hat{U}^\dagger \hat{a}_k \hat{U} = \cosh \mu \, \hat{a}_k + \sinh \mu \, \hat{a}_k^\dagger$ with $\mu \neq 0$. From the definitions (23), (24), we can calculate that r transforms to $r' = \cosh^2 \mu \, r + f(\mu) \neq r$, where $f(\mu)$ depends solely on correlations not included in the RSF formalism. Notably, the reduced entropy of r' differs from that of r. Finally, we note that by construction, the reduced entropy provides an upper bound to the von Neumann entropy, another quality shared with the Wehrl entropy.

As seen, the reduced entropy resembles the Wehrl entropy more than the von Neumann entropy. To make this point even stronger, we will now construct a new entropy of RSF based on the Wehrl entropy and show that for the majority of states it has approximately the same value as the reduced entropy. In other words, despite being based on the quantum von Neumann entropy, the reduced entropy gives the same quantitative results as RSF entropy based on the semiclassical Wehrl entropy.

Making use of the maximum entropy principle, analogously to the case of the original reduced entropy, we derive the *reduced Wehrl entropy*,

$$s_w(\hat{\rho}) := \text{tr} \ln(r_\alpha + \mathbf{1}_N) + N. \tag{43}$$

See Appendix B for details.

Just like the reduced entropy (41) maximizes the von Neumann entropy for a fixed RSF, the reduced Wehrl entropy maximizes the Wehrl entropy for a fixed RSF. We note that it has similar qualitative properties to the original reduced entropy, e.g., it is invariant under the same unitary transformations and is minimized by coherent states.

More importantly, the two entropies can also be linked quantitatively.

Proposition 3. The following relation between the RSF entropies holds:

$$0 < s_w(\hat{\rho}) - s_v(\hat{\rho}) \leqslant N. \tag{44}$$

Proof. We begin with the left-hand side (l.h.s.) inequality. Rearranging Eq. (41) we obtain

$$s_v(\hat{\rho}) = \text{tr}\{r_\alpha[\ln(r_\alpha + \mathbf{1}_N) - \ln r_\alpha]\}$$
$$+ \text{tr} \ln(r_\alpha + \mathbf{1}_N). \tag{45}$$

By definition of the reduced Wehrl entropy, the second term is equal to $-N + s_w(\hat{\rho})$. In the first term, we apply the eigendecomposition $r_\alpha = \sum_{k=1}^N \lambda_k |k\rangle\langle k|$, where $\lambda_k \geqslant 0$. Using basic properties of the logarithm, we arrive at

$$s_v(\hat{\rho}) = \sum_{k=1}^N \ln\left(1 + 1/\lambda_k\right)^{\lambda_k} - N + s_w(\hat{\rho}). \tag{46}$$

Clearly, the first term is maximized in the limit $\lambda_k \to \infty$, in which, by definition of the Euler's number, it approaches N. Then, the first and second terms cancel, leaving $s_v(\hat{\rho}) < s_w(\hat{\rho})$, as in the l.h.s. inequality.

To prove the r.h.s. inequality, we observe that, since $r_\alpha \geqslant 0$:

$$s_v(\hat{\rho}) \geqslant \text{tr}[(r_\alpha + \mathbf{1}_N)\ln(r_\alpha + \mathbf{1}_N)$$
$$- r_\alpha \ln(r_\alpha + \mathbf{1}_N)] = s_w(\hat{\rho}) - N, \tag{47}$$

which is equivalent to the r.h.s. inequality. ∎

Crucially, for states with mean particle number much bigger than the effective number of modes $\text{tr} r = \langle \hat{n} \rangle \gg N$, the term N is vanishing in comparison to s_w, s_v. Therefore, it follows from Eq. (44) that for most many-particle states, the two reduced entropies are effectively equal. Combining this with our previous analysis of the qualitative aspects of the two entropies, we see that in the RSF formalism, entropic descriptions based on the "quantum" von Neumann and on the "classical" Wehrl entropy are nearly identical to each other and akin to the Wehrl entropy [57]. This cements the classicality of the RSF description.

V. REDUCED KINETIC EQUATIONS

Having established the classicality of RSF, we can use the framework to derive and characterize the classical subset of Gaussian evolution. To do this, we will employ the final component of the formalism—reduced kinetic equations—which we summarize in this section.

Let us go back to the GKLS equation (10) and consider its general, nondiagonal form:

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_{k,k'} B_{kk'}\left(\hat{J}_k \hat{\rho} \hat{J}_{k'}^\dagger - \frac{1}{2}\{\hat{J}_{k'}^\dagger \hat{J}_k, \hat{\rho}\}\right). \tag{48}$$

Here, $\hat{J}_k$ are the jump operators and B is a nonnegative matrix. By diagonalizing the non-Hamiltonian part, the original equation (10) is obtained.

In the formalism of RSF, it is assumed that to correctly describe the dynamics of a macroscopic field, it is enough to treat it as a set of individual particles subject to spontaneous decay and production, as well as interaction with coherent classical sources and random scattering by the environment.

In such a setting, the following Hamiltonian arises [22]:

$$\hat{H} = \sum_{k=1}^{N} (\omega_k \hat{a}_k^\dagger \hat{a}_k + i\zeta_k \hat{a}_k^\dagger - i\zeta_k^* \hat{a}_k). \tag{49}$$

Here, the first term is the base Hamiltonian for bosonic fields, with the positive frequencies ω_k defining the energy levels of the system. The remaining two terms coincide with the Hermitian generator of the displacement operator (40) with argument $\vec{\zeta}$, which is experimentally realized by combining the input state with an N-mode coherent state $|\vec{\zeta}\rangle$ in an asymmetric beam-splitter [58]. Hence, this part of the Hamiltonian can be interpreted as an interaction with a coherent classical source with complex amplitudes ζ_k.

As for the dissipator, three families of jump operators were considered:

(1) $\hat{J}_k = \hat{a}_k$, that describe spontaneous decay of particles in the field at rates given by matrix $B_{kk'} = \Gamma_\downarrow^{k'k}$,

(2) $\hat{J}_k = \hat{a}_k^\dagger$, that describe spontaneous production of particles in the field at rates given by matrix $B_{kk'} = \Gamma_\uparrow^{kk'}$,

(3) $\hat{J}_k = \hat{U}_k$, with $\hat{U}_k$ being a unitary operator. For a large number of unitary operators, this family describes random scattering [35–38] with rates given by $B_{kk'} = \eta_k \delta_{kk'}$, $\eta_k \geqslant 0$, $\sum_k \eta_k = 1$ [59].

Note that, while it is not explicitly stated in the original work [22], the results stated there imply that the unitaries $\hat{U}_k$ must each transform the annihilation operators as

$$\hat{U}_k^\dagger \hat{a}_m \hat{U}_k = \sum_{l=1}^{N} (u_k)_{ml} \hat{a}_l, \tag{50}$$

where u_k have to be unitary to preserve the canonical commutation relations (19).

Under the collective influence of all these phenomena, the evolution of the density operator reads [22]

$$\frac{d}{dt}\hat{\rho} = -i\sum_{k=1}^{N} \omega_k [\hat{a}_k^\dagger \hat{a}_k, \hat{\rho}] + \sum_{k=1}^{N} [\zeta_k \hat{a}_k^\dagger - \zeta_k^* \hat{a}_k, \hat{\rho}]$$
$$+ \sum_{k,k'=1}^{N} \Gamma_\downarrow^{k'k} \left(\hat{a}_k \hat{\rho}\, \hat{a}_{k'}^\dagger - \frac{1}{2}\{\hat{a}_{k'}^\dagger \hat{a}_k, \hat{\rho}\} \right)$$
$$+ \sum_{k,k'=1}^{N} \Gamma_\uparrow^{kk'} \left(\hat{a}_k^\dagger \hat{\rho}\, \hat{a}_{k'} - \frac{1}{2}\{\hat{a}_{k'} \hat{a}_k^\dagger, \hat{\rho}\} \right)$$
$$+ \sum_k \eta_k \left(\hat{U}_k \hat{\rho}\, \hat{U}_k^\dagger - \hat{\rho} \right). \tag{51}$$

Note that the number of unitaries $\hat{U}_k$ is arbitrary.

Tracing both sides of Eq. (51) with $\hat{a}_l^\dagger \hat{a}_l$ and $\hat{a}_l$ yields the *reduced kinetic equations* for RSF. As the resulting equations slightly differ from the ones derived originally in Ref. [22], where minor errors appear [60], we provide them in full in the following proposition, with proof in Appendix C.

Proposition 4. The time evolution of RSF is governed by the *reduced kinetic equations*:

$$\frac{d}{dt}r = -i[h, r] + |\zeta\rangle\langle\alpha| + |\alpha\rangle\langle\zeta|$$
$$+ \frac{1}{2}\{\gamma_\uparrow - \gamma_\downarrow, r\} + \gamma_\uparrow$$
$$+ \sum_k \eta_k (u_k r u_k^\dagger - r),$$
$$\frac{d}{dt}|\alpha\rangle = -ih|\alpha\rangle + \frac{1}{2}(\gamma_\uparrow - \gamma_\downarrow)|\alpha\rangle + |\zeta\rangle$$
$$+ \sum_k \eta_k (u_k - 1)|\alpha\rangle. \tag{52}$$

Here,

$$h := \sum_{k=1}^{N} \omega_k |k\rangle\langle k|, \tag{53}$$

$$|\zeta\rangle := \sum_{k=1}^{N} \zeta_k |k\rangle, \tag{54}$$

$$\gamma_\updownarrow := \sum_{k,k'=1}^{N} \Gamma_\updownarrow^{kk'} |k\rangle\langle k'| \tag{55}$$

are the single-particle counterparts to $\hat{H}$, $\vec{\zeta}$, and $\Gamma_\updownarrow$, respectively, while u_k are fixed by Eq. (50).

The assumptions behind the model are best justified by its applicability. In the original work [22], the reduced kinetic equations were successfully used to describe macroscopic fields in thermal environments, as well as polarization optics, notably making an explicit connection between the Mueller and Jones calculi.

VI. CLASSICALITY OF QUANTUM GAUSSIAN EVOLUTION

Due to their full compatibility with RSF, the reduced kinetic equations necessarily constitute a semiclassical model of evolution. In this section, we use them as a tool for identifying the semiclassical subset of quantum Gaussian evolution, fulfilling the main goal of our work. For clarity, we consider each of the three terms entering the quantum Gaussian evolution equations (14) separately. All proofs are contained in Appendix D.

We begin with the Hamiltonian term.

Proposition 5. Let $(V, |\xi\rangle)$ denote the symplectic description of a system undergoing Gaussian Hamiltonian evolution

$$\frac{d}{dt}V = F_G(V), \quad \frac{d}{dt}|\xi\rangle = f_G(|\xi\rangle) \tag{56}$$

as given by Eq. (15). The evolution can be written as reduced kinetic equations (52) and is thus classical with respect to the RSF formalism if and only if

$$0 = [J, G]. \tag{57}$$

The corresponding reduced kinetic equations are governed by

$$h = i\mathcal{R}JG\mathcal{R}^\dagger, \tag{58}$$

with the remaining terms vanishing.

Proof. See Appendix D. ∎

To see what the condition (57) means, we make use of the matrix representation of the symplectic form (9). Substituting it into Eq. (57), we compute that the allowed Hamiltonians consist of 2×2 block matrices of the form

$$G_{kk'} = G_{k'k}^T = a_{kk'} \mathbf{1}_2 + (1 - \delta_{kk'}) b_{kk'} J_2, \qquad (59)$$

where k, k' enumerate the blocks and $a_{kk'}, b_{kk'} \in \mathbb{R}$. Making use of Eq. (11) we check that Eq. (59) allows only for particle number-preserving or *passive* interactions.

In standard optical implementations, passive transformations correspond to experimental operations with classical analogues, such as beam splitters and phase shifters. According to standard notions of nonclassicality, such as nonpositivity of the Glauber P representation or the presence of entanglement, the output of passive transformations can be nonclassical only if given nonclassical input [61,62]. The remaining *active* transformations, such as squeezing, have no classical analogues. Moreover, they can be a source of quantum advantage, e.g., in metrology [63,64]. Such transformations are forbidden by Eq. (59).

In short, Gaussian Hamiltonians are classical with respect to the reduced kinetic equations if they correspond to passive transformations.

Let us illustrate this with an example. Among the key ingredients in the resource-based approach to quantum thermodynamics are thermal operations, defined as energy-preserving operations on continuous variable systems coupled to a thermal environment. Due to the prevalence of quadratic Hamiltonians in experimental setups, special emphasis is put on *Gaussian thermal operations* (GTOs), which are thermal operations that preserve the set of Gaussian states.

Recently, a complete characterization of GTOs has been provided in Ref. [65]. Here, we focus on a natural subclass of GTOs generated by time-independent, nondegenerate Hamiltonians. Such GTOs are effectively reduced to single-mode transformations [65] of the form

$$V(t) = S[Q(t)S^{-1}V(0)(S^{-1})^T Q^T(t) + P]S^T, \qquad (60)$$

where S is a 2×2 symplectic matrix, $P := (1 - p)\nu \mathbf{1}$, and

$$Q(t) := \sqrt{p} \begin{bmatrix} \cos \phi(t) & \sin \phi(t) \\ -\sin \phi(t) & \cos \phi(t) \end{bmatrix}. \qquad (61)$$

Here, $\nu := \coth \beta \omega / 2$, ω is the Hamiltonian eigenvalue associated with the considered mode, β is the inverse temperature, while $p \in [0, 1]$.

Taking the time derivative of Eq. (60) we get

$$G = \frac{d\phi}{dt} J S S^T J^T, \qquad (62)$$

which, according to Proposition 6, governs classical evolution if

$$0 = [J, S S^T]. \qquad (63)$$

One can easily solve this condition explicitly, from which we find that S must be orthogonal in addition to being symplectic, meaning that it is passive [65].

Similar considerations concern Gaussian dissipative evolution based on Lindblad operators linear in mode quadratures.

Proposition 6. Let $(V, |\xi\rangle)$ denote the symplectic description of a system undergoing Gaussian dissipative evolution stemming from Lindblad operators linear in mode quadratures (12):

$$\frac{d}{dt}V = F_L(V), \qquad \frac{d}{dt}|\xi\rangle = f_L(|\xi\rangle) \qquad (64)$$

as given by Eq. (16). The evolution can be written as reduced kinetic equations (52) and is thus classical with respect to the RSF formalism if and only if

$$0 = [J, I_C] \qquad (65)$$

and

$$\gamma_\uparrow = \mathcal{R}(I_C J - J R_C J)\mathcal{R}^\dagger \geqslant 0, \qquad (66)$$

$$\gamma_\downarrow = -\mathcal{R}(I_C J + J R_C J)\mathcal{R}^\dagger \geqslant 0. \qquad (67)$$

The corresponding reduced kinetic equations are governed by $\gamma_\updownarrow$ as above with the remaining terms vanishing.

Proof. See Appendix D. ∎

Through Eq. (66) we can see that the matrix I_C describes the difference between particle creation and annihilation rates, i.e., particle flow: $\mathcal{R}I_C J \mathcal{R}^\dagger = \gamma_\uparrow - \gamma_\downarrow$. Thus, the first condition (65), by full analogy to the one for the Hamiltonian (57) means that the particle flow operator has to be passive. The second condition (66) simply requires nonnegative particle creation and annihilation rates.

As an example, let us consider *stabilizability* in two-mode entangled Gaussian systems. In quantum open systems, it is sometimes desirable to counteract the effects of dissipation by using an appropriate Hamiltonian. In the framework of stabilizability, one can check whether this is possible by solving a finite set of conditions rather than checking every Hamiltonian separately [41,66].

Recently, stabilizability was used to investigate the robustness of two-mode Gaussian states against three classes of dissipation [67]:

(1) local damping: $\hat{L}_k := \hat{a}_k$,
(2) global damping: $\hat{L} := (\hat{a}_1 + \hat{a}_2)$,
(3) dissipators engineered to preserve two-mode squeezed states:

$$\hat{L}_1 := \cosh \chi \, \hat{a}_1 - \sinh \chi \, \hat{a}_2^\dagger,$$
$$\hat{L}_2 := \cosh \chi \, \hat{a}_2 - \sinh \chi \, \hat{a}_1^\dagger, \qquad (68)$$

where $\chi \geqslant 0$ denotes the amount of squeezing.

It is straightforward to check that while all the dissipators fulfill Eq. (65), only the first two fulfill the positivity condition (66), unless no squeezing is considered in the third model ($\alpha = 0$). This, of course, makes sense from the point of classicality, since squeezing is a purely quantum resource, while the Lindblad operators appearing in the first two models merely describe particle loss in the system.

In addition, we remark that in the first and third models, the maximum amount of entanglement was stabilized in the system when using the Hamiltonian

$$\hat{H}_{sq} := -i\omega(\hat{a}_1 \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_2^\dagger), \qquad (69)$$

while in the second model the entanglement-maximizing Hamiltonian read

$$\hat{H} = \hat{H}_{\text{cas}} := -\frac{i\omega}{2}[(\hat{a}_1 + \hat{a}_2)^2 - (\hat{a}_1^\dagger + \hat{a}_2^\dagger)^2]. \quad (70)$$

As expected, neither Hamiltonian fulfills the classicality condition (57).

Finally, we consider Gaussian dissipative evolution based on unitary Lindblad operators.

Proposition 7. Let $(V, |\xi\rangle)$ denote the symplectic description of a system undergoing quantum Gaussian dissipative evolution stemming from unitary Lindblad operators (13):

$$\frac{d}{dt}V = F_U(V), \quad \frac{d}{dt}|\xi\rangle = f_U(|\xi\rangle) \quad (71)$$

as given by Eq. (17). The evolution can be written as reduced kinetic equations (52) and is thus classical with respect to the RSF formalism if and only if each K_j fulfills

$$0 = \mathcal{R}K_j\mathcal{R}^T \quad \text{and} \quad \mathcal{R}K_j\mathcal{R}^\dagger \text{ is unitary.} \quad (72)$$

The corresponding reduced kinetic equations are governed by

$$u_j = \mathcal{R}K_j\mathcal{R}^\dagger, \quad \eta_j = \gamma_j \quad (73)$$

with the remaining terms vanishing.

Proof. See Appendix D. ∎

Similar to previous results, the condition (72) is fulfilled only when the summation is over operations K_j, which are orthogonal in addition to being symplectic. From the physical point of view, they also correspond to passive transformations only [65].

Once again, we illustrate our result with an example. Let us consider the family of two-mode symplectic transformations $K_j = \exp[JS_j]$ generated by

$$S_j = w_j\begin{bmatrix} 0 & O_j \\ O_j & 0 \end{bmatrix}, \quad O_j = \begin{bmatrix} \cos\phi_j & \sin\phi_j \\ \sin\phi_j & -\cos\phi_j \end{bmatrix}, \quad (74)$$

where $w_j \geqslant 0$, $\phi_j \in [0, 2\pi)$. For $\phi = \pi/2$, K_j coincide with a transformation used for creation of highly entangled mixtures of Gaussian states in the asymptotic time limit in Ref. [35]. We can easily calculate that for all j, $\mathcal{R}K_j\mathcal{R}^\dagger = \cosh(w_j)\mathbf{1}_2$, which is unitary only in the trivial case $w_j = 0$. Thus, according to Proposition 7, the evolution is not classical, as expected given its entangling properties.

VII. CONCLUDING REMARKS

We studied the classicality of quantum Gaussian evolution, a model of time evolution relevant especially in modern quantum optics and continuous variables-based information processing. We derived an explicit set of conditions under which the evolution is classical, as summarized in Propositions 5–7 in Sec. VI. The derived conditions forbid Hamiltonians and Lindblad operators corresponding to so-called active optical transformations, such as squeezing, instead allowing only passive transformations, which have an intuitive experimental interpretation in terms of operations treating macroscopic light as a classical wave. Our results were obtained using the recent mesoscopic formalism of the reduced state of the field (RSF), which we redeveloped as a tool for classical description of many-particle bosonic fields.

Based on our findings, we suggest the following directions for further research. To start with, our investigations into the RSF framework could be generalized. For example, it would be interesting to see if our conjecture regarding lack of entanglement description via RSF can be proved (or disproved). Furthermore, the RSF formalism is based on one- and two-point correlation functions. Can a self-consistent mesoscopic framework based on higher-order correlations be designed? If so, then what new insights does it offer, in particular, with respect to classicality?

APPENDIX A: DERIVATION OF REDUCTION MAP (26)

In this Appendix, we derive the relation (26) between the RSF and covariance matrix pictures (52).

Beginning with the definition of the single-particle density matrix (23) and the annihilation and creation operators (18) we quickly obtain

$$r_{kk'} = \tfrac{1}{2}\text{Tr}[\hat{\rho}(\hat{x}_{k'}\hat{x}_k + i\hat{x}_{k'}\hat{p}_k - i\hat{p}_{k'}\hat{x}_k + \hat{p}_{k'}\hat{p}_k)]. \quad (A1)$$

Looking at Eqs. (1) and (5), we can see that

$$V_{2k'-1,2k-1} = V_{2k-1,2k'-1} = \text{Tr}[\hat{\rho}\,\hat{x}_{k'}\hat{x}_k],$$

$$V_{2k'-1,2k} = V_{2k,2k'-1} = \frac{1}{2}\text{Tr}[\hat{\rho}(\hat{x}_{k'}\hat{p}_k + \hat{p}_k\hat{x}_{k'})],$$

$$= \text{Tr}[\hat{\rho}\,\hat{x}_{k'}\hat{p}_k] - \frac{i}{2}\delta_{kk'},$$

$$V_{2k',2k-1} = V_{2k-1,2k'} = \frac{1}{2}\text{Tr}[\hat{\rho}(\hat{p}_{k'}\hat{x}_k + \hat{x}_k\hat{p}_{k'})],$$

$$= \text{Tr}[\hat{\rho}\,\hat{p}_{k'}\hat{x}_k] + \frac{i}{2}\delta_{kk'},$$

$$V_{2k',2k} = V_{2k,2k'} = \text{Tr}[\hat{\rho}\,\hat{p}_{k'}\hat{p}_k], \quad (A2)$$

where we made use of the canonical commutation relations (2). Substituting this into Eq. (A1), we quickly find that it is equivalent to the relation between r and V in Eq. (26). The relation between $|\alpha\rangle$ and $|\xi\rangle$ is derived in an analogous fashion.

The Heisenberg uncertainty principle (28) is derived by acting on the original Eq. (7) from the left with $\mathcal{R}$ and from the right with $\mathcal{R}^\dagger$, and using the easy-to-derive identity $\mathcal{R}J\mathcal{R}^\dagger = -i\mathbf{1}_N$, along with the previously derived Eq. (26).

APPENDIX B: DERIVATION OF REDUCED WEHRL ENTROPY

In this Appendix, we derive the reduced Wehrl entropy (43), defined as the maximum Wehrl entropy among all the states with a fixed RSF.

First, let us observe that RSF has the following representation in terms of the Husimi Q function:

$$r_{kk'} = \int \frac{d^{2N}\vec{\beta}}{\pi^N}(\beta_k\beta_{k'}^* - \delta_{kk'})Q(\vec{\beta}),$$

$$\alpha_k = \int \frac{d^{2N}\vec{\beta}}{\pi^N}\beta_k Q(\vec{\beta}). \tag{B1}$$

The maximum Wehrl entropy among all the states with fixed RSF can then be found by finding the extremum of the following functional with respect to Q:

$$S_W[Q] - \lambda f[Q] - \sum_{k,k'=1}^{N}\mu_{k'k}g_{kk'}[Q]$$
$$+ \sum_{k=1}^{N}t_k^* h_k[Q] + \sum_{k=1}^{N}s_k h_k^*[Q], \tag{B2}$$

where S_W is the Wehrl entropy (42) and the three constraints

$$f[Q] := \int \frac{d^{2N}\vec{\beta}}{\pi^N}Q(\vec{\beta}) - 1 = 0,$$

$$g_{kk'}[Q] := \int \frac{d^{2N}\vec{\beta}}{\pi^N}(\beta_k\beta_{k'}^* - \delta_{kk'})Q(\vec{\beta}) - r_{kk'} = 0,$$

$$h_k[Q] := \int \frac{d^{2N}\vec{\beta}}{\pi^N}\beta_k Q(\vec{\beta}) - \alpha_k = 0 \tag{B3}$$

fix the normalization and the RSF of the state to $(r, |\alpha\rangle)$ [cf. Eq. (B1)]. Finally, λ, $\mu_{k'k}$, t_k, and s_k are the Lagrange multipliers. Note that the signs, as well as the notation (e.g., t_k^* instead of t_k) in Eq. (B2) are arbitrary. Therefore, we made a choice that anticipates the final result best.

The solution to the variational problem is given by

$$\tilde{Q}(\vec{\beta}) := Ae^{-\vec{\beta}^\dagger\mu\vec{\beta}+\vec{t}^\dagger\vec{\beta}+\vec{\beta}^\dagger\vec{s}}, \tag{B4}$$

where A is a normalization constant. Substituting the solution into the three constraints (B3) and making use of the integration formula [68]

$$\int \frac{d^{2N}\vec{\beta}}{\pi^N}e^{-\vec{\beta}^\dagger\mu\vec{\beta}+\vec{t}^\dagger\vec{\beta}+\vec{\beta}^\dagger\vec{s}} = \frac{1}{\det\mu}e^{\vec{t}^\dagger\mu^{-1}\vec{s}} \tag{B5}$$

yields

$$A = \det\mu\, e^{-\vec{t}^\dagger\mu^{-1}\vec{s}}, \quad \mu^{-1} = r_\alpha + \mathbf{1}_N, \quad \vec{t} = \vec{s} = \mu\vec{\alpha}, \tag{B6}$$

and in turn

$$\tilde{Q}(\vec{\beta}) = \frac{1}{\det(r_\alpha + \mathbf{1}_N)}e^{-(\vec{\beta}-\vec{\alpha})^\dagger(r_\alpha+\mathbf{1}_N)^{-1}(\vec{\beta}-\vec{\alpha})}. \tag{B7}$$

Plugging this into the definition of Wehrl entropy (42) leads to Eq. (43).

APPENDIX C: DERIVATION OF REDUCED KINETIC EQUATIONS

In this Appendix, we derive the reduced kinetic equations (52) from the GKLS equation for macroscopic fields (51).

By definition, the single-particle density matrix evolves as

$$\frac{d}{dt}r_{ll'} = \mathrm{Tr}\left(\frac{d}{dt}\hat{\rho}\,\hat{a}_{l'}^\dagger\hat{a}_l\right) = \sum_{n=1}^{5}(\Phi_n)_{ll'}, \tag{C1}$$

where [cf. Eq. (51)]

$$(\Phi_1)_{ll'} := -i\sum_{k=1}^{N}\omega_k\mathrm{Tr}([\hat{a}_k^\dagger\hat{a}_k, \hat{\rho}]\hat{a}_{l'}^\dagger\hat{a}_l), \tag{C2}$$

$$(\Phi_2)_{ll'} := \sum_{k=1}^{N}\mathrm{Tr}([\zeta_k\hat{a}_k^\dagger - \zeta_k^*\hat{a}_k, \hat{\rho}]\hat{a}_{l'}^\dagger\hat{a}_l), \tag{C3}$$

$$(\Phi_3)_{ll'} := \sum_{k,k'=1}^{N}\Gamma_\downarrow^{k'k}\mathrm{Tr}\left[\left(\hat{a}_k\hat{\rho}\,\hat{a}_{k'}^\dagger - \frac{1}{2}\{\hat{a}_{k'}^\dagger\hat{a}_k, \hat{\rho}\}\right)\hat{a}_{l'}^\dagger\hat{a}_l\right], \tag{C4}$$

$$(\Phi_4)_{ll'} := \sum_{k,k'=1}^{N}\Gamma_\uparrow^{kk'}\mathrm{Tr}\left[\left(\hat{a}_k^\dagger\hat{\rho}\,\hat{a}_{k'} - \frac{1}{2}\{\hat{a}_k\hat{a}_{k'}^\dagger, \hat{\rho}\}\right)\hat{a}_{l'}^\dagger\hat{a}_l\right], \tag{C5}$$

$$(\Phi_5)_{ll'} := \sum_k \eta_k\mathrm{Tr}[(\hat{U}_k\hat{\rho}\,\hat{U}_k^\dagger - \hat{\rho})\hat{a}_{l'}^\dagger\hat{a}_l]. \tag{C6}$$

Let us focus on the first term, Φ_1. From the cyclic property of the trace

$$(\Phi_1)_{ll'} = -i\sum_{k=1}^{N}\omega_k\mathrm{Tr}(\hat{\rho}[\hat{a}_{l'}^\dagger\hat{a}_l, \hat{a}_k^\dagger\hat{a}_k]). \tag{C7}$$

The commutator can be easily calculated with the use of the canonical commutation relations (19) and the well-known property

$$[\hat{O}_1\hat{O}_2, \hat{O}_3\hat{O}_4] = \hat{O}_1[\hat{O}_2, \hat{O}_3]\hat{O}_4 + [\hat{O}_1, \hat{O}_3]\hat{O}_2\hat{O}_4$$
$$+ \hat{O}_3\hat{O}_1[\hat{O}_2, \hat{O}_4] + \hat{O}_3[\hat{O}_1, \hat{O}_4]\hat{O}_2, \tag{C8}$$

valid for arbitrary $\hat{O}_j$.

We obtain

$$(\Phi_1)_{ll'} = -i(\omega_l - \omega_{l'})\mathrm{Tr}(\hat{\rho}\,\hat{a}_{l'}^\dagger\hat{a}_l). \tag{C9}$$

Using the definitions (23) and (53), it is easy to show that the above is equivalent to

$$(\Phi_1)_{ll'} = -i\sum_{j=1}^{N}(h_{lj}r_{jl'} - r_{lj}h_{jl'}), \tag{C10}$$

or simply

$$\Phi_1 = -i[h, r]. \tag{C11}$$

This shows that the first term on the r.h.s. of Eq. (51) transforms into the first term on the r.h.s. of Eq. (52).

In an analogous way, we can show that

$$\Phi_2 = |\zeta\rangle\langle\alpha| + |\alpha\rangle\langle\zeta|, \tag{C12}$$

$$\Phi_3 = -\tfrac{1}{2}\{\gamma_\downarrow, r\}, \tag{C13}$$

$$\Phi_4 = \tfrac{1}{2}\{\gamma_\uparrow, r\} + \gamma_\uparrow. \tag{C14}$$

As for Φ_5, due to normalization of η_k to one, the second term under the trace in Eq. (C6) gives rise to simply $r_{ll'}$. The first

term can be rewritten using the cyclic property of the trace and the fact that $\hat{U}_k \hat{U}_k^\dagger = \hat{\mathbf{1}}$:

$$(\Phi_5)_{ll'} = \sum_k \eta_k \mathrm{Tr}[\hat{\rho}\,(\hat{U}_k^\dagger \hat{a}_{l'}^\dagger \hat{U}_k)(\hat{U}_k^\dagger \hat{a}_l \hat{U}_k)] - r_{ll'}. \quad (\text{C15})$$

Making use of the assumption (50), we quickly find that

$$\Phi_5 = \sum_k \eta_k (u_k r u_k^\dagger - r). \quad (\text{C16})$$

This finishes the derivation of the reduced kinetic equation (52) for r. The corresponding equation for $|\alpha\rangle$ is derived in the same way.

APPENDIX D: DERIVATION OF CLASSICAL GAUSSIAN EVOLUTION

In this Appendix, we prove Propositions 5, 6, and 7, i.e., we derive the conditions under which Gaussian evolution is equivalent to the reduced kinetic equations.

To this end, it will be useful to define an auxiliary field, which we call *conjugate RSF*:

$$c := \sum_{k,k'=1}^N \mathrm{Tr}[\hat{\rho}\,\hat{a}_{k'}\hat{a}_k]|k\rangle\langle k'|,$$

$$|\alpha^*\rangle := \sum_{k=1}^N \mathrm{Tr}[\hat{\rho}\,\hat{a}_k^\dagger]|k\rangle. \quad (\text{D1})$$

Mirroring the derivation of the relation (26) between RSF and the symplectic picture, one can show that

$$c = \mathcal{R} V \mathcal{R}^T, \quad |\alpha^*\rangle = \mathcal{R}^* |\xi\rangle. \quad (\text{D2})$$

We will also make heavy use of the following property of the reduction matrix:

$$\mathcal{R}^\dagger \mathcal{R} = \tfrac{1}{2}(\mathbf{1} + iJ). \quad (\text{D3})$$

Notably,

$$\mathcal{R}^\dagger \mathcal{R} + \mathcal{R}^T \mathcal{R}^* = \mathbf{1}. \quad (\text{D4})$$

1. Proof of Proposition 5

We begin with the Hamiltonian evolution (56). Making extensive use of the identity (D4), along with relations (26) and (D2), we obtain the corresponding evolution equations for RSF:

$$\frac{d}{dt}r = y_G r - r y_G^\dagger + z_G c^\dagger + c z_G^\dagger + \frac{1}{2}(y_G - y_G^\dagger),$$

$$\frac{d}{dt}|\alpha\rangle = y_G|\alpha\rangle + z_G|\alpha^*\rangle, \quad (\text{D5})$$

where

$$y_G := \mathcal{R} J G \mathcal{R}^\dagger, \quad z_G := \mathcal{R} J G \mathcal{R}^T. \quad (\text{D6})$$

Unlike the reduced kinetic equations, this evolution equation for RSF couples it to the conjugate field. Therefore, if the two equations are to coincide for arbitrary input states, the c-dependent terms must vanish. This implies $z_G = 0$ and in turn $0 = \mathcal{R}^\dagger z_G \mathcal{R}^*$, which is equivalent to the condition (57), as we intended to show.

Under this condition y_G is Hermitian, and hence the final equations read

$$\frac{d}{dt}r = [y_G, r] \quad \frac{d}{dt}|\alpha\rangle = y_G|\alpha\rangle. \quad (\text{D7})$$

Clearly, they have the form of the reduced kinetic equations (52) with Eq. (58) at the input.

2. Proof of Proposition 6

In the case of the dissipative evolution stemming from linear Lindblad operators (64), using Eqs. (26), (D2), and (D4) as previously yields

$$\frac{d}{dt}r = y_L r + r y_L^\dagger + z_L c^\dagger + c z_L^\dagger + \frac{1}{2}(y_L + y_L^\dagger),$$

$$\frac{d}{dt}|\alpha\rangle T = y_L|\alpha\rangle + z_L|\alpha^*\rangle, \quad (\text{D8})$$

where

$$y_L := \mathcal{R} J I_C \mathcal{R}^\dagger, \quad z_L := \mathcal{R} J I_C \mathcal{R}^T, \quad w := \mathcal{R} J R_C J^T \mathcal{R}^\dagger. \quad (\text{D9})$$

Once again, we must require the equation to be c-independent. This implies $z_L = 0$ and in turn $0 = \mathcal{R}^\dagger z_L \mathcal{R}^*$, which is the same as the condition (65) that we wanted to derive.

Under this condition y_L is Hermitian, and hence the final equations read

$$\frac{d}{dt}r = \{y_L, r\} + y_L + w, \quad \frac{d}{dt}|\alpha\rangle = y_L|\alpha\rangle. \quad (\text{D10})$$

It is not difficult to show that these equations have the form of the reduced kinetic equations (52) with Eq. (66) at the input. Note that for this identification to have a physical meaning, the particle creation and annihilation rates have to be nonnegative.

3. Proof of Proposition 7

Finally, we consider the dissipative evolution stemming from unitary Lindblad operators (71). Once again making use of Eqs. (26), (D2), and (D4) we obtain

$$\frac{d}{dt}r = \sum_j \gamma_j \bigg[q_j r q_j^\dagger + s_j r^T s_j^\dagger - r + q_j c s_j^\dagger + s_j c^* q_j^\dagger,$$

$$+ \frac{1}{2}(q_j q_j^\dagger + s_j s_j^\dagger - \mathbf{1}) \bigg],$$

$$\frac{d}{dt}|\alpha\rangle = \sum_j \gamma_j [(q_j - \mathbf{1})|\alpha\rangle + s_j|\alpha^*\rangle], \quad (\text{D11})$$

where

$$q_j = \mathcal{R} K_j \mathcal{R}^\dagger, \quad s_j = \mathcal{R} K_j \mathcal{R}^T. \quad (\text{D12})$$

Calculating analogously as in the previous cases, we find that the equation is c-independent if for all j

$$0 = \mathcal{R}K_j\mathcal{R}^T. \tag{D13}$$

To get a correspondence with the reduced kinetic equations, we must additionally require all q_j to be unitary. The two conditions are collectively captured by Eq. (72), finishing the proof.

The final equations read

$$\frac{d}{dt}r = \sum_j \gamma_j(q_j r q_j^\dagger - r),$$

$$\frac{d}{dt}|\alpha\rangle = \sum_j \gamma_j(q_j|\alpha\rangle - |\alpha\rangle), \tag{D14}$$

which have the form of the reduced kinetic equations (52) with Eq. (73) at the input.

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Classicality of the Bogoliubov Transformations and the Dynamical Casimir Effect Through the Reduced State of the Field

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We use the reduced state of the field formalism (*Entropy* **21**, 705 (2019)) to derive conditions under which a Bogoliubov transformation can be considered semi-classical. We apply this result to the dynamical Casimir effect in a moving medium (*Phys. Rev. A* **78**, 042109 (2008)), discussing its classical and quantum features.

topics: Bogoliubov transformations, dynamical Casimir effect, quantum-to-classical transitions, mesoscopic degrees of freedom

1. Introduction

Arguably one of the most surprising predictions of quantum field theory is the Casimir effect, a physical force arising solely from the presence of quantum fluctuations in the vacuum [1–3]. Since its original formulation in 1948 [4], the phenomenon has garnered a lot of interest, in particular giving rise to many alternative formulations and generalizations. One such generalization, dubbed the dynamical Casimir effect, predicts the spontaneous production of particles in a medium following from non-trivial time dependence of either its boundary or its material coefficients [5–8].

In 2008, Professor Iwo Białynicki-Birula working together with Professor Zofia Białynicka-Birula[†] established a third mechanism generating the dynamical Casimir effect — oscillatory motion of

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[†]This is a good opportunity to acknowledge the fact that 59 papers out of a total of 206 so far published by Professor Białynicki-Birula, as well as the comprehensive textbook on quantum electrodynamics [9], have been written in this admirable collaboration which started as early as 1957 [10].

a medium [9–11]. In fact, this mechanism is more general and applies to all kinds of motion, as long as its speed varies in time, and one carefully picks the "incoming" and "outgoing" annihilation and creation operators (see an example of a uniformly accelerated medium [12]). A loosely related phenomenon occurs around large rotating and/or gravitating bodies [13].

The dynamical Casimir effect is obtained by performing a Bogoliubov transformation, i.e., a linear transformation of the creation and annihilation operators of the quantum field preserving canonical commutation relations [14]. If the Casimir effects are among the most interesting phenomena in quantum theory, Bogoliubov transformations are among its most reliable tools. Originally used to describe superconductivity [15, 16], today they are widely used in many branches of quantum physics, from optics and theories of magnetism to field theory in a curved spacetime (Unruh effect, Hawking radiation) [14, 17–19].

While the most prominent applications of the Bogoliubov transformations suggest the latter to be inherently quantum, we observe that from the formal point of view, Bogoliubov transformations are

essentially equivalent to a change of basis of the Hilbert space. For this reason, one may expect that at least some Bogoliubov transformations could have classical analogs, similar to local unitary rotations of the Hilbert space, which do not entangle the system. If so, this could shed new conceptual light on the phenomena described by them.

In this paper, we derive an exact set of conditions under which Bogoliubov transformations can be considered semi-classical. By semi-classical (further also referred to as just "classical"), we understand models which can be described by certain kinetic equations for reduced single-particle states and their displacements — so-called *reduced state of the field* formalism [20]. This framework has recently been proven to be an effective tool in probing the classicality of quantum Gaussian evolution [21].

In the case of isolated systems, the transformations allowed by our conditions turn out to have a simple interpretation in terms of passive operations, which correspond to classical devices such as beam splitters. In the case of open systems, the conditions are less restrictive, which we interpret as some of the total dynamics' "quantumness" being encoded into the environment. Our findings allow us to conduct an in-depth discussion of the classicality of the dynamical Casimir effect derived in [11]. We find that, while the overall phenomenon is quantum in nature, the individual photons experience each other as semi-classical dissipative effects.

This paper is organized as follows. In Sect. 2, we introduce the dynamical Casimir effect in moving media. In Sect. 3, we briefly summarize the most important properties of our main tool — the reduced state of the field (RSF). In Sect. 4, we derive our main results, namely classicality conditions for Bogoliubov transformations. In Sect. 5, we build upon these findings to assess the classicality of the dynamical Casimir effect. We conclude in Sect. 6.

2. Dynamical Casimir effect in a moving medium

The electromagnetic field is fully described by the set of four three-component vectors, D and E, describing the electric field, along with B and H, describing the magnetic field, which altogether fulfill the Maxwell equations in vacuum [22, 23]

$$\partial_t D(r, t) = \nabla \times H(r, t),$$

$$\nabla \cdot D(r, t) = 0,$$

$$-\partial_t B(r, t) = \nabla \times E(r, t),$$

$$\nabla \cdot B(r, t) = 0. \tag{1}$$

In the Heisenberg picture, the operators associated with these fields fulfill exactly the same set of equations.

Assuming the field propagates through a homogeneous, isotropic medium moving with a velocity v and characterized by constant material coefficients μ, ϵ, the field vectors are related by the Minkowski constitutive relations [24]

$$D + \frac{v}{c^2} \times H = \epsilon \left(E + v \times B \right),$$

$$B - \frac{v}{c^2} \times E = \mu \left(H - v \times D \right), \tag{2}$$

where c is the speed of light.

In the convenient Riemann–Silberstein approach (see a review [25]), the electromagnetic field is combined into two vectors

$$F := \frac{1}{\sqrt{2}\,\epsilon} D + \frac{\mathrm{i}}{\sqrt{2}\,\mu} B,$$

$$G := \frac{1}{\sqrt{2}\,\mu} E + \frac{\mathrm{i}}{\sqrt{2}\,\epsilon} H. \tag{3}$$

The advantage of this approach can already be seen in the considered problem, as the constitutive relations (2) can always be solved for G, yielding

$$G = \frac{c}{n} \left[F + \frac{n^2 - 1}{c^2 n^2 - v^2}\, v \times \left(v \times F + \mathrm{i}\, c n F \right) \right], \tag{4}$$

where $n := c\sqrt{\epsilon\mu} \geqslant 1$ is the refractive index of the medium. Then, assuming position-independent velocity, $v(r, t) = c\beta(t)$, the vacuum Maxwell equations (1) reduce to just one equation

$$\partial_t F = -\mathrm{i}\, c\, \delta(t) \left(\beta(t) \cdot \nabla \right) F + \frac{c}{n} \alpha(t)\, \nabla \times F$$

$$-\frac{c}{n}\delta(t)\, \beta(t) \times \nabla \left(\beta(t) \cdot F \right), \tag{5}$$

where

$$\delta(t) := \frac{n^2 - 1}{n^2 - \beta^2(t)}, \qquad \alpha(t) := 1 - \delta(t)\beta^2(t). \tag{6}$$

Under a further assumption that the velocity has a constant direction m, and with the help of the Fourier decomposition

$$F(r, t) = \int \frac{\mathrm{d}^3 k}{\sqrt{(2\pi)^3}}\, \mathrm{e}^{\mathrm{i}\, k \cdot r - \mathrm{i}\, \phi(k, t)}$$

$$\times \left[e(k) f_+(k, t) + e^*(k) f_-(k, t) \right], \tag{7}$$

where e are elliptic polarization vectors [11], the Maxwell equations lead to a pair of ordinary differential equations for the functions $f_\pm$

$$\partial_t \hat{f}_\pm(k, t) =$$

$$\mp\, \mathrm{i}\omega(k) \left[\eta_+(k, t)\hat{f}_\pm(k, t) - \eta_-(k, t)\hat{f}_\mp(k, t) \right], \tag{8}$$

with

$$\eta_\pm(k, t) := \frac{1}{2} \left[\frac{\alpha(t)}{\sigma^2(k)} \pm \sigma^2(k)\Delta(k, t) \right],$$

$$\Delta(k, t) := 1 - \delta(t)\beta^2(t) \cos^2\left(\theta(k) \right). \tag{9}$$

The parameter θ denotes the angle between the wave vector $\boldsymbol{k}$ and the velocity direction $\boldsymbol{m}$, while σ is a *free* real parameter defining the polarization geometry. Last but not least, the phase

$$\phi(\boldsymbol{k}, t) := \omega(\boldsymbol{k}) \cos\left(\theta(\boldsymbol{k})\right) \int_0^t \mathrm{d}\tau\, \delta(\tau)\beta(\tau) \qquad (10)$$

has been extracted to achieve a simplification of the resulting equations ($\beta = |\boldsymbol{\beta}|$).

To obtain the dynamical Casimir effect, it is assumed that the medium is moving with a time-dependent velocity from time $t = 0$ up to $t = T$ [11, 12]. If the medium just before and after was "still" (characterized by $\beta(t) = \text{const}$), the corresponding operators $\hat{f}_{\pm}$, after a suitable choice of σ [11], can be interpreted in terms of the creation and annihilation operators of photons with right helicity

$$\hat{f}_+(\boldsymbol{k}, t) = \begin{cases} \sqrt{\hbar\omega(\boldsymbol{k})}\, \hat{a}_{R,\text{in}}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\,\omega(\boldsymbol{k})t}, & t < 0 \\[2mm] \sqrt{\hbar\omega(\boldsymbol{k})}\, \hat{a}_{R,\text{out}}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\,\omega(\boldsymbol{k})(t-T)}, & t > T \end{cases}, \qquad (11)$$

and left helicity

$$\hat{f}_-(\boldsymbol{k}, t) = \begin{cases} \sqrt{\hbar\omega(\boldsymbol{k})}\, \hat{a}_{L,\text{in}}^{\dagger}(-\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\omega(\boldsymbol{k})t}, & t < 0 \\[2mm] \sqrt{\hbar\omega(\boldsymbol{k})}\, \hat{a}_{L,\text{out}}^{\dagger}(-\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\omega(\boldsymbol{k})(t-T)}, & t > T \end{cases}. \qquad (12)$$

Here, $\hat{a}_{L/R,\text{in/out}}$ and their Hermitian conjugates fulfill all the expected properties of the standard annihilation and creation operators. Note that such interpretation is not possible during the acceleration period $t \in [0, T]$ itself, due to the impossibility of separation into positive and negative frequency parts.

The final operators are given by the initial ones via the relation [11]

$$\hat{a}_{R,\text{out}}(\boldsymbol{k}) = \mathrm{e}^{-\mathrm{i}\,\phi}\left[f_{R+}\hat{a}_{R,\text{in}}(\boldsymbol{k}) + f_{R-}\hat{a}_{L,\text{in}}^{\dagger}(-\boldsymbol{k})\right],$$

$$\hat{a}_{L,\text{out}}^{\dagger}(-\boldsymbol{k}) = \mathrm{e}^{-\mathrm{i}\,\phi}\left[f_{L+}\hat{a}_{R,\text{in}}(\boldsymbol{k}) + f_{L-}\hat{a}_{L,\text{in}}^{\dagger}(-\boldsymbol{k})\right], \qquad (13)$$

where $\phi \equiv \phi(\boldsymbol{k}, T)$, while $f_{L\pm} \equiv f_{L\pm}(\boldsymbol{k}, T)$, $f_{R\pm} \equiv f_{R\pm}(\boldsymbol{k}, T)$ are solutions to the differential equations (8) subject to initial conditions

$$f_{R+}(\boldsymbol{k}, 0) = f_{L-}(\boldsymbol{k}, 0) = 1,$$

$$f_{R-}(\boldsymbol{k}, 0) = f_{L+}(\boldsymbol{k}, 0) = 0. \qquad (14)$$

It is worth adding that, due to the canonical commutation relations for the outgoing photons (13)

$$\left[\hat{a}_{R,\text{out}}(\boldsymbol{k}), \hat{a}_{R,\text{out}}^{\dagger}(\boldsymbol{k})\right] = 1,$$

$$\left[\hat{a}_{R,\text{out}}(\boldsymbol{k}), \hat{a}_{R,\text{out}}(\boldsymbol{k})\right] = 0, \qquad (15)$$

we have

$$|f_{R+}|^2 = |f_{R-}|^2 + 1, \qquad (16)$$

with an analogous relation for f_{L+} and f_{L-}.

Let us remark that in the original work [11], the functions f were denoted as $f_{\pm}^1 \equiv f_{R\pm}$ and $f_{\pm}^2 \equiv f_{L\pm}$. Here, we change the notation to make the connection to photon helicity more immediate, as well as to avoid confusing the indices with exponentiation. We stress, however, that despite corresponding to different photon helicities, the two pairs of functions are interrelated via the initial conditions and have to be considered together.

The Casimir effect is finally obtained by considering the system initially in the vacuum and computing the photon number densities after the motion

$$\langle \hat{n}_R(T) \rangle = \langle 0|\, \hat{a}_{R,\text{out}}^{\dagger}(\boldsymbol{k})\hat{a}_{R,\text{out}}(\boldsymbol{k})\,|0\rangle =$$
$$\left|f_{R-}(\boldsymbol{k}, T)\right|^2 \delta(0),$$

$$\langle \hat{n}_L(T) \rangle = \langle 0|\, \hat{a}_{L,\text{out}}^{\dagger}(\boldsymbol{k})\hat{a}_{L,\text{out}}(\boldsymbol{k})\,|0\rangle =$$
$$\left|f_{L+}(\boldsymbol{k}, T)\right|^2 \delta(0), \qquad (17)$$

where $\delta(0)$ is the Dirac delta singularity. Note that, due to the symmetry of the evolution equations governing the left and right helicity functions, the two densities are, in fact, equal

$$\langle \hat{n}_R(T) \rangle = \langle \hat{n}_L(T) \rangle \equiv \langle \hat{n}(T) \rangle. \qquad (18)$$

As was verified in [11, 12], at least for some $\boldsymbol{k}$, this number is a growing function of T. Therefore, the motion of the medium results in potentially unbounded particle production in the vacuum and, hence, the prediction of the dynamical Casimir effect.

Transformation (13) at the heart of the discussed phenomenon is an example of a *Bogoliubov transformation* [15, 16], namely a linear transformation $\left\{\hat{a}_n, \hat{a}_n^{\dagger}\right\} \to \left\{\hat{a}_n', \hat{a}_n'^{\dagger}\right\}$ of the creation and annihilation operators preserving the canonical commutation relations [14]. As the main result of this paper, we will derive the precise conditions under which such transformations can be considered semiclassical, with special emphasis put on the classicality of the dynamical Casimir effect in a moving medium.

3. Reduced state of the field

To assess the (semi)classicality of Bogoliubov transformations, we first need to define a sensible criterion for what is classical. To this end, we will employ the mesoscopic formalism of the reduced state of the field (RSF) [20], which was already used for similar purposes before [21]. Since the framework itself is not the main focus of our study, here we provide only basic information about it. For more details, see the introduction of the formalism by Robert Alicki in [20], its semi-classical interpretation in [21], and its application to thermodynamics in [26].

We consider an N-mode, continuous variable Hilbert space described by a set of N annihilation and creation operators $\hat{a}_k$, $\hat{a}_{k'}^{\dagger}$ fulfilling the

canonical commutation relations

$$\left[\hat{a}_k, \hat{a}^\dagger_{k'}\right] = \delta_{kk'}, \qquad \left[\hat{a}_k, \hat{a}_{k'}\right] = \left[\hat{a}^\dagger_k, \hat{a}^\dagger_{k'}\right] = 0. \tag{19}$$

As always, an arbitrary n-particle state in the many-body Hilbert space can be constructed by acting on the vacuum state with n appropriate creation operators. Since, in principle, the number of particles in a given mode can be arbitrary, the N-mode Hilbert space is infinitely dimensional, and so is the density operator $\hat{\rho}$ constituting the full quantum description of the system.

In some cases, however, the full quantum formalism is not necessary and can be replaced by a simpler, mesoscopic framework. For example, Gaussian states and dynamics can be efficiently studied in the symplectic picture [27–29]. Similarly, to describe macroscopic fields and associated evolution, a formalism called *reduced state of the field* (RSF) has been recently developed [20].

In the RSF framework, instead of by the density operator, the system is described by the pair $(r, |\alpha\rangle)$. Here,

$$r := \sum_{k,k'=1}^N \operatorname{Tr}\left[\hat{\rho}\,\hat{a}^\dagger_{k'}\hat{a}_k\right] |k\rangle \langle k'| \tag{20}$$

is the *single-particle density matrix*, while the *averaged field* equals

$$|\alpha\rangle := \sum_{k=1}^N \operatorname{Tr}\left[\hat{\rho}\,\hat{a}_k\right] |k\rangle. \tag{21}$$

The single-particle density matrix contains the simplest non-local information about the system. Additionally, its diagonal elements equal the mean particle numbers $r_{kk} = \langle \hat{a}^\dagger_k \hat{a}_k \rangle$. Consequently, the matrix is normalized to the mean total particle number. Note that, by construction, the single-particle density matrix is non-negative. The averaged field, on the other hand, contains additional local information.

Much like the previously mentioned symplectic picture requires observables and transformations that are Gaussian, the RSF formalism employs observables that are either *additive* [20]

$$\hat{O} = \sum_{k,k'=1}^N o_{kk'}\,\hat{a}^\dagger_k \hat{a}_{k'}, \tag{22}$$

or *linear*

$$\hat{\sigma} = \sum_{k=1}^N \left(\sigma^*_k \hat{a}_k + \sigma_k \hat{a}^\dagger_k\right). \tag{23}$$

In the case of macroscopic fields, which are usually modeled as non-interacting fields with dynamics governed by equations linear in creation and annihilation operators, the most relevant observables are of this form. For example, the Hamiltonian is additive, while the position and momentum operators are linear.

Defining the *reduced observables* corresponding to (22) and (23) as

$$o = \sum_{k,k'=1}^N o_{kk'} |k\rangle \langle k'|, \qquad |\sigma\rangle = \sum_{k=1}^N \sigma_k |k\rangle, \tag{24}$$

we can indeed see that the associated expectation values can be rewritten in the RSF formalism as [21]

$$\operatorname{Tr}\left[\hat{\rho}\,\hat{O}\right] = \operatorname{tr}(r\,o), \qquad \operatorname{Tr}\left[\hat{\rho}\,\hat{\sigma}\right] = \langle\sigma|\alpha\rangle + \langle\alpha|\sigma\rangle. \tag{25}$$

The RSF framework comes equipped with dedicated entropy measures and evolution equations, both derived from the standard quantum description. In the case of entropy, we have the reduced von Neumann and Wehrl entropies [20, 21]

$$s_v(r, |\alpha\rangle) := \operatorname{tr}\left[(r_\alpha + \mathbb{1}_N)\ln(r_\alpha + \mathbb{1}_N) - r_\alpha \ln(r_\alpha)\right],$$

$$s_w(r, |\alpha\rangle) := \operatorname{tr}\left[\ln(r_\alpha + \mathbb{1}_N)\right] + N, \tag{26}$$

where $r_\alpha := r - |\alpha\rangle\langle\alpha|$ and $\mathbb{1}_N$ denotes the identity matrix in dimension N. The reduced entropies arise from applying the maximum entropy principle to the standard von Neumann and Wehrl entropies, respectively [30, 31].

Finally, RSF evolves according to the *reduced kinetic equations* [20, 21]

$$\frac{dr}{dt} = -\frac{i}{\hbar}[h, r] + |\zeta\rangle\langle\alpha| + |\alpha\rangle\langle\zeta|$$
$$+ \frac{1}{2}\{\gamma_\uparrow - \gamma_\downarrow, r\} + \gamma_\uparrow + \sum_j \eta_j\left(u_j r u^\dagger_j - r\right),$$

$$\frac{d|\alpha\rangle}{dt} = -\frac{i}{\hbar}h|\alpha\rangle + \frac{1}{2}\left(\gamma_\uparrow - \gamma_\downarrow\right)|\alpha\rangle + |\zeta\rangle$$
$$+ \sum_j \eta_j\left(u_j - 1\right)|\alpha\rangle, \tag{27}$$

which are derived from the Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) equation [32, 33] under the assumption that the considered quantum field can be treated as a set of individual particles subject to spontaneous decay and production, as well as interaction with coherent classical sources and random scattering by the environment. The operators entering (27) represent

- The Hamiltonian

$$h := \hbar \sum_{k=1}^N \omega_k |k\rangle\langle k|, \quad \omega_k \geqslant 0; \tag{28}$$

- Coherent sources

$$|\zeta\rangle := \sum_{k=1}^N \zeta_k |k\rangle; \tag{29}$$

- Particle creation rates

$$\gamma_\uparrow = \sum_{k,k'=1}^N \gamma^{kk'}_\uparrow |k\rangle\langle k'|, \qquad \gamma_\uparrow \geqslant 0, \tag{30}$$

  and analogously particle annihilation rates $\gamma_\downarrow$;

- Unitary interactions with rates $\eta_j \geqslant 0$ ($\sum_j \eta_j = 1$)

$$u_j = \sum_{k,k'=1}^N u^{kk'}_j |k\rangle\langle k'|, \qquad u^\dagger_j u_j = u_j u^\dagger_j = \mathbb{1}_N. \tag{31}$$

For a large number of non-commuting unitaries, the last term in either of the reduced kinetic equations represents random scattering.

Note that, while not explicitly stated in the original work [20], it is clear from the derivation of the reduced kinetic equations that all the quantities entering it may be time-dependent, provided they fulfill the respective constraints (imposed by complete positivity) during every instant of the evolution.

Although RSF was originally designed to capture the quantum features of macroscopic fields, it has been recently shown to have a convincing interpretation as a semi-classical description of bosonic many-body systems [21]. For example, it was proved that the RSF formalism contains no information about distillable entanglement in the system, and that both of the reduced entropies are akin to Wehrl's semi-classical entropy [31], typically considered as such due to its close association with the phase-space.

Because, by construction, the reduced kinetic equations (27) preserve the RSF formalism's semi-classical set of degrees of freedom, any time evolution model of the density operator, which can be rewritten as reduced kinetic equations, must be necessarily semi-classical itself. Based on this principle, in [21], quantum Gaussian evolution of light was found to be classical if and only if it consisted strictly of so-called passive optical transformations, e.g., beam splitting and phase shifting. Contrary to their remaining active counterparts, such as quantum squeezing, passive transformations can be understood operationally by treating light as a classical wave. In this paper, we adopt a similar methodology for Bogoliubov transformations — if they preserve the set of the degrees of freedom contained within RSF, we will regard them as semi-classical, and if not, we will regard them as inherently quantum.

4. Classicality of Bogoliubov transformations

We are now equipped with the tools necessary to assess the classicality of Bogoliubov transformations. We will consider two distinct cases, i.e., Bogoliubov transformations in isolated (closed) systems and in open systems. The main results of this section are presented in Propositions 1–3, with proofs in Appendix A.

4.1. Isolated system

In the case of an isolated system, the most general transformation of the density operator is unitary

$$\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger. \tag{32}$$

For the transformation to be of the Bogoliubov-type, $\hat{U}$ must be such that, for some complex matrix $\mathcal{X}$,

$$\hat{A}'_n := \hat{U}^\dagger \hat{A}_n \hat{U} = \sum_{m=1}^{2N} \mathcal{X}_{nm}\hat{A}_m, \tag{33}$$

with

$$\hat{A}_n := \begin{cases} \hat{a}_n, & n \in \{1, \dots, N\}, \\ \hat{a}_n^\dagger, & n \in \{N+1, \dots, 2N\}. \end{cases} \tag{34}$$

To preserve the canonical commutation relations, the matrix $\mathcal{X}$ has to fulfill the so-called symplectic property [34, 35]

$$\mathcal{X}\mathcal{S}\mathcal{X}^\dagger = \mathcal{S}, \tag{35}$$

where $\mathcal{S} = \mathrm{diag}\big[\mathbb{1}_N, -\mathbb{1}_N\big]$. As a consequence of the symplectic property,

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_\uparrow & \mathcal{X}_\downarrow \\ \mathcal{X}_\downarrow^* & \mathcal{X}_\uparrow^* \end{bmatrix}, \tag{36}$$

where $\mathcal{X}_\uparrow^\downarrow$ are of size $N \times N$.

Calculating the change in RSF implied by a generic Bogoliubov transformation and forcing the result to be fully contained within the formalism, we obtain the classicality conditions for the closed system Bogoliubov transformations. Furthermore, if the unitary transformation in (32) depends smoothly on time, then so does the matrix $\mathcal{X}$, turning the discrete Bogoliubov transformation into a continuous *Bogoliubov evolution*. In such a case, the density operator can be differentiated with respect to time, and the resulting evolution equation can be compared with the reduced kinetic equations.

Proceeding in this way, we obtain our first major result.

Proposition 1. Isolated system Bogoliubov transformations (as described above) are compatible with the RSF formalism and are thus classical with respect to it if and only if

$$0 = \mathcal{X}_\downarrow. \tag{37}$$

Additionally, if the transformation depends smoothly on time, the corresponding reduced kinetic equations (27) exist and are governed by

$$h = \frac{i\hbar}{2}\left(\frac{d\mathcal{X}_\uparrow}{dt}\mathcal{X}_\uparrow^{-1} - \mathcal{X}_\uparrow^{-\dagger}\frac{d\mathcal{X}_\uparrow^\dagger}{dt}\right), \tag{38}$$

with the remaining terms vanishing.

Proof. See Appendix A. □

The obtained classicality condition is easy to interpret. Substituting (37) into the symplectic condition (35), we immediately find that $\mathcal{X}$ is also unitary in addition to being symplectic, which means that it is passive. Thus, in a complete analogy to quantum Gaussian evolution [21], Bogoliubov transformations in isolated systems are semi-classical only if they correspond to passive transformations.

Let us also remark that while the absence of the dissipative terms in the obtained reduced kinetic equations was to be expected in an isolated system, the lack of coherent classical sources was not. Indeed, it is easy to see that this lack is not

a fundamental property of the Bogoliubov evolution, but rather a consequence of the Bogoliubov transformations (33) being defined, for simplicity, without constant terms (independent of the creation and annihilation operators).

4.2. Open system

In the more general case of an open system, the total density operator of the system and environment (also called bath) is as well transformed according to (32). However, we are only interested in the state of the system, given by a partial trace over the degrees of freedom of the environment:

$$\hat{\rho}_S = \mathrm{Tr}_E[\,\hat{\rho}\,]. \qquad (39)$$

The Bogoliubov transformation itself (33) remains the same. Still, assuming the system and the environment span N_S and N_E modes respectively, it is convenient to additionally split the matrices entering the block decomposition (36) into

$$\mathcal{X}_\uparrow = \begin{bmatrix} \mathcal{X}_{\uparrow S} & \mathcal{X}_{\uparrow C} \\ \mathcal{X}_{\uparrow C'} & \mathcal{X}_{\uparrow E} \end{bmatrix}, \qquad \mathcal{X}_\downarrow = \begin{bmatrix} \mathcal{X}_{\downarrow S} & \mathcal{X}_{\downarrow C} \\ \mathcal{X}_{\downarrow C'} & \mathcal{X}_{\downarrow E} \end{bmatrix}, \qquad (40)$$

where $\mathcal{X}_{\updownarrow S}$ is an $N_S \times N_S$ matrix associated with the system, $\mathcal{X}_{\updownarrow E}$ is an $N_E \times N_E$ matrix associated with the environment, and $\mathcal{X}_{\updownarrow C}$, $\mathcal{X}_{\updownarrow C'}$ are appropriately-sized matrices associated with both. Note that the case of the closed system can be retrieved easily by setting $N_E = 0$ (which, in particular, implies $\mathcal{X}_\updownarrow = \mathcal{X}_{\updownarrow S}$) and dropping the then-redundant lower indices S.

For a generic initial state of the bath-system ensemble, the dynamics of the latter cannot be separated from the dynamics of the former, making it impossible to even compare with the RSF formalism. Nonetheless, even in this completely general setting, we were able to derive necessary conditions for classicality of Bogoliubov transformations.

Proposition 2. Open system Bogoliubov transformations (as described above) can be compatible with the RSF formalism and thus be classical with respect to it only if

$$0 = \mathcal{X}_{\downarrow S}. \qquad (41)$$

Proof. See Appendix A. ☐

Unlike the condition (37) for the closed system, the classicality condition for the open system is difficult to interpret. However, comparing it with its closed system counterpart, we can at least see that the latter is much more restrictive: it requires the whole matrix $\mathcal{X}_\downarrow$ to vanish, while the former requires only its system part $\mathcal{X}_{\downarrow S}$ to vanish. Therefore, depending on how we define the degrees of freedom of the system, we may find the same total dynamics to be either classical or quantum from the point of view of the system. This will indeed be the case in

the next section, where we will find that the dynamical Casimir effect falls exactly into this category.

Still, any such interpretation has to be made with care, since it must be stressed that the condition (41) is not equivalent to classicality, but only necessary for it. In stark contrast to the closed system, in the case of an open system, whether or not a given Bogoliubov transformation is classical from the point of view of RSF depends not only on the matrix $\mathcal{X}$ defining it, but also on the total initial state of the system-environment ensemble. It is possible that, for particularly strongly correlated total initial states, the only semi-classical Bogoliubov transformations are those that induce completely separate dynamics for the system and environment, essentially defying the notion of an open system.

To make stronger statements, we are therefore forced to make some restrictions. Firstly, we assume that the initial total state is separable with respect to the bipartition between the system and the bath. This is a typical assumption in the theory of quantum open systems. In particular, the GKLS equation cannot be derived without it [36]. Since, in particular, the reduced kinetic equations governing the time evolution in the RSF formalism are derived from a GKLS equation, it is only natural to also make this assumption in the present case.

Secondly, we assume that the bath is initially in the vacuum state. Note that while this assumption is a very strong one, it is fulfilled by many well-studied and useful models, such as quantum-limited amplification, quantum-limited attenuation, and phase conjugation channels, utilized, e.g., in studies of Gaussianity, entropy, and entanglement [37–39]. More importantly for us, as we will discuss in the next section, it is also satisfied by the dynamical Casimir effect.

Under the above assumptions, we obtain our final main result for Bogoliubov transformations.

Proposition 3. The classicality condition (41) is both necessary and sufficient for open system Bogoliubov transformations with the environment initially in the vacuum state. Additionally, if such transformations depend smoothly on time, the corresponding reduced kinetic equations exist provided

$$\mathcal{W} \geqslant 0, \qquad \mathcal{W} - \mathcal{Y}_r \geqslant 0 \qquad (42)$$

and are governed by

$$h = -\frac{\hbar \mathcal{Y}_i}{2}, \quad \gamma_\downarrow = \mathcal{W}, \quad \gamma_\uparrow = \mathcal{W} - \mathcal{Y}_r, \qquad (43)$$

with the remaining terms vanishing. Here,

$$\begin{aligned}
\mathcal{Y}_i &:= -\mathrm{i}\left(\mathcal{Y} - \mathcal{Y}^\dagger\right), \\
\mathcal{Y} &:= \frac{\mathrm{d}\mathcal{X}_{\uparrow S}}{\mathrm{d}t}\mathcal{X}_{\uparrow S}^{-1}, \\
\mathcal{Y}_r &:= \mathcal{Y} + \mathcal{Y}^\dagger, \\
\mathcal{D} &:= \mathcal{X}_{\downarrow C}\mathcal{X}_{\downarrow C}^\dagger, \\
\mathcal{W} &:= \frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t} - \mathcal{Y}\mathcal{D} - \mathcal{D}\mathcal{Y}^\dagger.
\end{aligned} \qquad (44)$$

Proof. See Appendix A. ☐

Interestingly, the obtained Bogoliubov reduced kinetic equations do not depend on any components of the matrix $\mathcal{X}$ labeled by the subscripts C', despite depending on the components labeled by C. At first, this may appear surprising, since a priori both are equally responsible for describing the correlations between the system and the environment. The asymmetry is resolved by interpreting the C components as encoding the influence of the environment on the system, and the C' components as encoding the influence of the system on the environment. The lack of the C' components in the description of the system then becomes expected. As an additional argument for this view, we observe that if we exchanged the roles of the system and the environment, the equations would depend on the C' components, with the C components missing.

Proposition 3 will be our main tool in the study of the classicality of the dynamical Casimir effect. Before we do it, however, let us illustrate our results so far with a short but instructive example — the Gaussian amplification process.

Example (Gaussian amplification process). In the Gaussian amplification process, an arbitrary initial state of the N-mode system

$$\hat{\rho}(t_0) = \int \frac{\mathrm{d}^{2N} \boldsymbol{z}_0}{\pi^N} P_0(\boldsymbol{z}_0) |\boldsymbol{z}_0\rangle \langle \boldsymbol{z}_0| \qquad (45)$$

is driven by a heat bath into the state [40]

$$\hat{\rho}(t) = \int \frac{\mathrm{d}^{2N} \boldsymbol{z}_0}{\pi^N} P_0(\boldsymbol{z}_0) \bigotimes_{j=1}^{N} \int \frac{\mathrm{d}^2 z_j}{\pi} \rho_j(t) |z_j\rangle \langle z_j|,$$

$$\rho_j(t) := \frac{1}{n_j(t)} \exp\left(-\frac{|z_j - z_{0j} \mathrm{e}^{\kappa_j t}|^2}{n_j(t)}\right). \qquad (46)$$

Here, the integration is over the real and imaginary parts of the complex vectors $\boldsymbol{z}_0$, $\boldsymbol{z}$; $P_0(\boldsymbol{z}_0)$ denotes the Glauber–Sudarshan P representation [41, 42] of the initial state; $|z_j\rangle$ are coherent states; κ_j is the amplification rate of the j-th mode; and

$$n_j(t) := (1 + m_j)\left(\mathrm{e}^{2\kappa_j t} - 1\right), \qquad (47)$$

where m_j is the mean number of photons in the j-th mode of the bath, assumed to be effectively constant throughout the whole process (this is true as long as the bath is much bigger than the system).

The corresponding RSF can be easily calculated

$$r_{kk'}(t) = \int \frac{\mathrm{d}^{2N} \boldsymbol{z}_0}{\pi^N} P_0(\boldsymbol{z}_0) \prod_{j=1}^{N} \int \frac{\mathrm{d}^2 z_j}{\pi} \rho_j(t) z_k z_{k'}^*,$$

$$\alpha_k(t) = \int \frac{\mathrm{d}^{2N} \boldsymbol{z}_0}{\pi^N} P_0(\boldsymbol{z}_0) \prod_{j=1}^{N} \int \frac{\mathrm{d}^2 z_j}{\pi} \rho_j(t) z_k. \qquad (48)$$

The integrals over z_j can be performed using the standard result [43]

$$\int \frac{\mathrm{d}^{2N} \boldsymbol{z}}{\pi^N} \mathrm{e}^{-\boldsymbol{z}^\dagger \mu \boldsymbol{z} + \boldsymbol{s}^\dagger \boldsymbol{z} + \boldsymbol{z}^\dagger \boldsymbol{s}} = \frac{\mathrm{e}^{\boldsymbol{s}^\dagger \mu^{-1} \boldsymbol{s}}}{\det[\, \mu \,]}, \qquad (49)$$

where μ denotes an invertible matrix and $\boldsymbol{s}$ is a vector of size N. In our case,

$$\mu^{-1} = n(t) := \sum_{j=1}^{N} n_j(t) |j\rangle \langle j|,$$

$$\boldsymbol{s} = n^{-1}(t) |\boldsymbol{z}_0(t)\rangle,$$

$$|\boldsymbol{z}_0(t)\rangle := \sum_{j=1}^{N} z_{0j} \mathrm{e}^{\kappa_j t} |j\rangle. \qquad (50)$$

This yields

$$r(t) = n(t) + \big\langle\, |\boldsymbol{z}_0(t)\rangle\langle\boldsymbol{z}_0(t)| \,\big\rangle_0,$$

$$|\alpha(t)\rangle = \big\langle\, |\boldsymbol{z}_0(t)\rangle \,\big\rangle_0, \qquad (51)$$

where $\langle \cdot \rangle_0 := (\pi^{-N}) \int \mathrm{d}^{2N} \boldsymbol{z}_0 \, P_0(\boldsymbol{z}_0)(\cdot)$. The formulae (51) induce the following differential evolution equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2}\Big\{ 2\kappa\,(\mathbb{1} + m) - 2\kappa m, r \Big\} + 2\kappa\,(\mathbb{1} + m),$$

$$\frac{\mathrm{d}\,|\alpha\rangle}{\mathrm{d}t} = \frac{1}{2}\Big(2\kappa\,(\mathbb{1} + m) - 2\kappa m \Big) |\alpha\rangle, \qquad (52)$$

where $m := \sum_{j=1}^{N} m_j |j\rangle \langle j|$ and $\kappa := \sum_{j=1}^{N} \kappa_j |j\rangle \langle j|$. Clearly, the equations have the form of reduced kinetic equations (27) with $\gamma_\uparrow = 2\kappa\,(\mathbb{1} + m)$, $\gamma_\downarrow = 2\kappa m$ and $h = |\zeta\rangle = \mu(\mathrm{d}u) = 0$.

According to Proposition 2, any open system Bogoliubov evolution that can be represented by reduced kinetic equations has to necessarily fulfill the classicality condition (41). To see that this is indeed the case in the Gaussian amplification process, we observe that it is generated by a Bogoliubov transformation of the form [37]

$$\mathcal{X}_\uparrow = \cosh\,(\kappa t) \begin{bmatrix} \mathbb{1}_N & 0 \\ 0 & \mathbb{1}_N \end{bmatrix},$$

$$\mathcal{X}_\downarrow = \sinh\,(\kappa t) \begin{bmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{bmatrix}. \qquad (53)$$

Clearly, $\mathcal{X}_{\downarrow S}$, being the upper left-hand side block component of $\mathcal{X}_\downarrow$, vanishes, as required by the aforementioned condition.

The fact that we found the Gaussian amplification process to be semi-classical is not surprising — intuitively, Gaussian amplification can be interpreted as pumping particles into the system until it reaches essentially macroscopic size. The process is well known for turning quantum phenomena into more classical ones. For example, it was previously shown that the Glauber-Sudarshan P distribution of an infinitely amplified state approaches the semi-classical Husimi Q distribution [44, 45]. Similarly, the von Neumann entropy of the maximally amplified state approaches the semi-classical Wehrl entropy [30, 38]. More recently, it has been shown that the amplified Pegg-Barnett phase formalism approaches the Paul phase formalism [46].

5. Classicality of the dynamical Casimir effect

Armed with the classicality conditions (37) and (41), we are now ready to come back to the dynamical Casimir effect. We begin by observing that while the phenomenon spans an infinite number of modes of photons with both helicities, its defining Bogoliubov transformation (13) couples them in pairs only. Any mode $\boldsymbol{k}$ of the right helicity photons is coupled only to itself and the mode $-\boldsymbol{k}$ of the left helicity photons. For this reason, we can restrict our analysis to two modes with no loss in generality.

Written in terms of the matrix $\mathcal{X}$, the Bogoliubov transformation (13) reads

$$
\mathcal{X} = \begin{bmatrix}
\mathrm{e}^{-\mathrm{i}\phi} f_{R+} & 0 & 0 & \mathrm{e}^{-\mathrm{i}\phi} f_{R-} \\
0 & \mathrm{e}^{\mathrm{i}\phi} f_{L-}^{*} & \mathrm{e}^{\mathrm{i}\phi} f_{L+}^{*} & 0 \\
0 & \mathrm{e}^{\mathrm{i}\phi} f_{R-}^{*} & \mathrm{e}^{\mathrm{i}\phi} f_{R+}^{*} & 0 \\
\mathrm{e}^{-\mathrm{i}\phi} f_{L+} & 0 & 0 & \mathrm{e}^{-\mathrm{i}\phi} f_{L-}
\end{bmatrix}.
$$

$$(54)$$

The interpretation of classicality depends on what we consider to be the system.

In the most natural view, the system spans photons with both left and right helicity. Hence, we have a closed, two-mode system. Comparing (54) with (36), we easily find the classicality criterion (37) to read explicitly

$$
f_{R-}(\boldsymbol{k}, T) = 0 = f_{L+}(\boldsymbol{k}, T). \tag{55}
$$

Looking at (17), we can immediately see that this implies no Casimir effect, i.e., the photon production in the vacuum is zero. Thus, according to the RSF formalism, any dynamical Casimir effect is necessarily non-classical, as expected.

To see the physical reason for this, we go back to the differential equations (8), along with the initial conditions (14). It is easy to see that (8) can be fulfilled if and only if $\eta_{-}(\boldsymbol{k}, t) = 0$. This is equivalent to $\sigma(\boldsymbol{k}) = [\alpha/\Delta(\boldsymbol{k})]^{1/4}$, where, due to the time-independence of σ, α and Δ have to be time-independent too, implying constant velocity. The equations for the remaining functions can then be easily solved, yielding [11]

$$
f_{R+}(\boldsymbol{k}, t) = f_{L-}^{*}(\boldsymbol{k}, t) = \mathrm{e}^{-\mathrm{i}\tilde{\omega}(\boldsymbol{k})t}, \tag{56}
$$

where $\tilde{\omega} = \omega\sqrt{\alpha\Delta}$. Substituting this into (13), we find that the final creation and annihilation operators simplify to just

$$
\hat{a}_{R,\mathrm{out}}(\boldsymbol{k}) = \mathrm{e}^{-\mathrm{i}[\phi(\boldsymbol{k},T)+\tilde{\omega}(\boldsymbol{k})T]} \, \hat{a}_{R,\mathrm{in}}(\boldsymbol{k}),
$$

$$
\hat{a}_{L,\mathrm{out}}^{\dagger}(-\boldsymbol{k}) = \mathrm{e}^{-\mathrm{i}[\phi(\boldsymbol{k},T)-\tilde{\omega}(\boldsymbol{k})T]} \, \hat{a}_{L,\mathrm{in}}^{\dagger}(-\boldsymbol{k}),
$$

$$(57)$$

i.e., they are multiplied by a phase. Obviously, this phase is irrelevant to the expectation values of the corresponding number operators on the vacuum, which is why the dynamical Casimir effect cannot take place for constant velocities.

However, there is another point of view. Nothing stops us from interpreting exclusively the left helicity photons as the system, and the right helicity photons as the environment. Then, we are dealing with an open one-mode system subject to influence from a one-mode environment. By comparing (54) with (36), (40), we immediately find that now, the classicality condition (41) always holds, regardless of the form of the functions $f_{R\pm}$, $f_{L\pm}$. Crucially, because the mode associated with the right helicity photons is initially in the vacuum state, then, due to Proposition 3, this classicality condition is both necessary and sufficient. Does this mean that the Casimir effect is, in the end, classical? Or maybe it means that the RSF formalism is not a valid tool for probing classicality after all?

In our opinion, neither. Consider, for example, the maximally entangled two-qubit Bell state [47, 48]

$$
|\Phi_{+}\rangle := \tfrac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big). \tag{58}
$$

If, in an analogy to the Casimir effect, we consider only the first qubit as the system, we will find it to be in the maximally mixed state

$$
\hat{\rho}_{S} = \mathrm{Tr}_{\mathrm{2nd\ qubit}} |\Phi_{+}\rangle\langle\Phi_{+}| = \tfrac{1}{2}\hat{\mathbb{1}}_{2}, \tag{59}
$$

which can certainly be considered classical. Of course, this does not mean that the Bell state that we started with was classical. Instead, its "quantumness" was contained in the correlations between the two qubits, rather than any of the two qubits themselves.

In the case of the Casimir effect and the Bogoliubov transformations in general, it is even more apparent what happens with the quantumness. Consider the matrix element $\mathcal{X}_{\downarrow 12} = \mathcal{X}_{\downarrow C} = f_{R-}(\boldsymbol{k}, T)$, which in our case, encodes the correlations between photons with left and right helicities. For a generic initial state, these correlations are potentially quantum. Thus, if a closed system is to be considered classical, they must necessarily vanish $\mathcal{X}_{\downarrow 12}=\mathcal{X}_{\downarrow C}=0$, as they constitute an integral part of the system. However, in the case of an open system, the discussed correlations are no longer part of the system, and instead enter it only at the level of the environmental effects, most easily seen through the evolution (44). Therefore, even if they have a strictly quantum origin, the system experiences them only as dissipation, which in this case happens to have a semi-classical interpretation in terms of particle annihilation and creation rates.

Alternatively, we can think of the Casimir process as consisting of two parts. The first, captured by the matrix $\mathcal{X}_{\uparrow}$, describes the morphing of photons with left helicity into those with right helicity and vice versa. The second, captured by the matrix $\mathcal{X}_{\downarrow}$, describes the creation of photons with both helicities. The former, being semi-classical, is unconstrained by the RSF formalism. The latter, however, being more quantum in nature, is forbidden by RSF, unless the quantumness can be encoded into the environment, as discussed previously.

Finally, let us observe that even though the Bogoliubov transformation (13) is technically of the discrete type, as the creation and annihilation operators are formally ill-defined during the acceleration period $t \in [0, T]$, the functions $f_{R\pm}$ and $f_{L\pm}$ defining the transformation are well defined at all times. Adding that to the fact that the final moment of acceleration T is completely arbitrary, we can consider (13) as defining a smooth Bogoliubov evolution in the parameter T.

Since, as explained previously, the initial total state fulfills the requirements of Proposition 3, the Bogoliubov evolution at hand must have a representation in terms of the reduced kinetic equations (27) with (43) at the input. Indeed, making use of the latter equation, we find

$$h = \hbar\omega \left(\eta_+ + \eta_- \, \mathrm{Re} \left[\frac{f_{R-}}{f_{R+}} \right] + \delta\beta \cos(\theta) \right),$$

$$\gamma_\uparrow = 2\omega\eta_- \frac{|f_{R-}|^2}{|f_{R+}|^2} \, \mathrm{Im} \left[\frac{f_{R+}}{f_{R-}} \right],$$

$$\gamma_\downarrow = 0. \tag{60}$$

For more details regarding the derivation of these identities, see Appendix B. Here, we focus on their physical significance.

To start with, we note that, as expected, the Hamiltonian for the photons is proportional to their frequency. Furthermore, the particle annihilation rate is zero, which intuitively corresponds to the fact that the dynamical Casimir effect results only in the spontaneous creation of particles, not their disappearance. Finally, once again abusing the differential equations (8), we can easily calculate that the time derivative of the total photon density (18) equals

$$\frac{\mathrm{d}\langle \hat{n} \rangle}{\mathrm{d}T} = 2\omega\eta_- |f_{R-}|^2 \, \mathrm{Im} \left[\frac{f_{R+}}{f_{R-}} \right], \tag{61}$$

which, using (16) and (60), can be rewritten as simply

$$\frac{\mathrm{d}\langle \hat{n} \rangle}{\mathrm{d}T} = \gamma_\uparrow \left(\langle \hat{n} \rangle + 1 \right). \tag{62}$$

This result has three worthwhile implications.

Firstly, it has a sound physical interpretation: the time derivative of the total photon density in the dynamical Casimir effect turns out to be simply proportional to the current photon density times the current particle creation rate. Secondly, it tells us that the non-negativity of $\gamma_\uparrow$, which is required for the result to be physical, is equivalent to the non-negativity of photon number growth. In particular, because of the initial condition (14), a valid matrix $\gamma_\uparrow$ by its very construction prevents negative photon numbers. Finally, because of the $\langle \hat{n} \rangle$-independent term on the r.h.s., our final result (62) proves that the dynamical Casimir effect occurs for any non-zero $\gamma_\uparrow$, which can be traced to any non-constant velocity of the medium ($\gamma_\uparrow = 0$ holds only for $\eta_- = 0$, which holds only for $\boldsymbol{\beta} = \mathrm{const}$).

6. Conclusions

In this paper, we employed the recent mesoscopic formalism of the reduced state of the field to derive the exact conditions under which Bogoliubov transformations in either isolated or open systems should be considered semi-classical. Applying our result to the case of the dynamical Casimir effect in the medium moving with a varying speed, we found that, while the photons with left and right helicity see each other as semi-classical objects, the Casimir effect itself is genuinely quantum, as expected. Let us stress that the analysis is made possible because for each wave vector, we can consider two polarization degrees of freedom. Therefore, it is essential that the described phenomenon is "*based on full Maxwell equations in three dimensions*" as pointed out at the end of the Conclusions section in [11].

Appendix A

In this appendix, we prove our main results regarding the classicality of Bogoliubov transformations, i.e., Propositions 1–3.

To this end, in addition to RSF, we will employ two auxiliary mesoscopic fields. The first, defined originally in [21], is the *conjugate RSF*

$$c := \sum_{k,k'=1}^{N} \mathrm{Tr} \left[\hat{\rho} \, \hat{a}_{k'} \hat{a}_k \right] |k\rangle \langle k'| ,$$

$$|\alpha^*\rangle := \sum_{k=1}^{N} \mathrm{Tr} \left[\hat{\rho} \, \hat{a}_k^\dagger \right] |k\rangle . \tag{63}$$

The second is the *generalized RSF*

$$g := \sum_{k,k'=1}^{2N} \mathrm{Tr} \left[\hat{\rho} \, \hat{A}_{k'}^\dagger \hat{A}_k \right] |k\rangle \langle k'| ,$$

$$|\mathcal{A}\rangle := \sum_{k=1}^{2N} \mathrm{Tr} \left[\hat{\rho} \, \hat{A}_k \right] |k\rangle . \tag{64}$$

It is easy to see that the three reduced fields are related to each other as follows

$$g = \begin{bmatrix} r & c \\ c^* & r^T + \mathbb{1}_N \end{bmatrix},$$

$$|\mathcal{A}\rangle = |\alpha\rangle \oplus |\alpha^*\rangle . \tag{65}$$

We add that, by definition, $r = r^\dagger$, $c = c^T$, and $|\alpha\rangle^* = |\alpha^*\rangle$.

Proof of Proposition 1

We start with Proposition 1. It is easy to see that due to (64), (32), (33), under a generic Bogoliubov transformation, the generalized RSF $(g, |\mathcal{A}\rangle)$ transforms as

$$g' = \mathcal{X} g \mathcal{X}^\dagger, \qquad |\mathcal{A}'\rangle = \mathcal{X} |\mathcal{A}\rangle. \qquad (66)$$

Then, (65) and (36) imply

$$r' := \mathcal{X}_\uparrow r \mathcal{X}_\uparrow^\dagger + \mathcal{X}_\uparrow c \mathcal{X}_\downarrow^\dagger + \mathcal{X}_\downarrow c^\dagger \mathcal{X}_\uparrow^\dagger + \mathcal{X}_\downarrow \left(r_S^T + \mathbb{1}_N \right) \mathcal{X}_\downarrow^\dagger,$$

$$|\alpha'\rangle = \mathcal{X}_\uparrow |\alpha\rangle + \mathcal{X}_\downarrow |\alpha^*\rangle. \qquad (67)$$

Clearly, this couples RSF to the conjugate field, meaning that it does not preserve the set of the associated degrees of freedom. For an arbitrary initial state, the coupling vanishes only if (37) is fulfilled, which is what we wanted to show.

Assuming the time-dependent case with the classicality condition (37) fulfilled, (67) reduces to

$$r(t) = \mathcal{X}_\uparrow(t) r(t_0) \mathcal{X}_\uparrow^\dagger(t),$$

$$|\alpha(t)\rangle = \mathcal{X}_\uparrow(t) |\alpha(t_0)\rangle. \qquad (68)$$

These equations are reversible, i.e.,

$$r(t_0) = \mathcal{X}_\uparrow^{-1}(t) r(t) \mathcal{X}_\uparrow^{-\dagger}(t),$$

$$|\alpha(t_0)\rangle = \mathcal{X}_\uparrow^{-1}(t) |\alpha(t)\rangle. \qquad (69)$$

Taking the time derivative of (68) and making use of (69), we obtain the reduced kinetic equations (27) with (38) at the input. This concludes the proof.

Proof of Proposition 2

To prove Proposition 2, we observe that the reduced fields of the total state of the system and the environment have the structure

$$r = \begin{bmatrix} r_S & r_C \\ r_C^\dagger & r_E \end{bmatrix}, \qquad c = \begin{bmatrix} c_S & c_C \\ c_C^T & c_E \end{bmatrix},$$

$$|\alpha\rangle = |\alpha^*\rangle^* = |\alpha_S\rangle \oplus |\alpha_E\rangle, \qquad (70)$$

where $(r_S, |\alpha_S\rangle)$, $(c_S, |\alpha_S^*\rangle)$ are the reduced fields of the system; $(r_E, |\alpha_E\rangle)$, $(c_E, |\alpha_E^*\rangle)$ are the reduced fields of the environment; and r_C, c_C contain the system-bath correlations. This fact follows directly from the definitions of the fields. For example,

$$(r_S)_{kk'} := \text{Tr}\left[\text{Tr}_E(\hat{\rho}) \hat{a}_{k'}^\dagger \hat{a}_k \right] = \text{Tr}\left[\hat{\rho}\, \hat{a}_{k'}^\dagger \hat{a}_k \right] := r_{kk'}. \qquad (71)$$

The remaining relations are proved in a similar fashion.

For a generic initial total state, the dynamics are quite complex. Making use of the block-form decompositions (70) and (40) in (68), we obtain a rather lengthy expression for the transformed RSF of the system, which can be written as

$$r_S' = F_{\uparrow\uparrow}(r) + F_{\downarrow\uparrow}(c^*) + F_{\uparrow\downarrow}(c) + F_{\downarrow\downarrow}\left(r^T + 1 \right),$$

$$|\alpha_S'\rangle = \mathcal{X}_{\uparrow S} |\alpha_S\rangle + \mathcal{X}_{\uparrow C} |\alpha_E\rangle + \mathcal{X}_{\downarrow S} |\alpha_S^*\rangle + \mathcal{X}_{\downarrow C} |\alpha_E^*\rangle, \qquad (72)$$

where

$$F_{ab}(x) := \mathcal{X}_{aS} x_S \mathcal{X}_{bS}^\dagger + \mathcal{X}_{aS} x_C \mathcal{X}_{bC}^\dagger + \mathcal{X}_{aC} x_C^\dagger \mathcal{X}_{bS}^\dagger$$

$$+ \mathcal{X}_{aC} x_E \mathcal{X}_{bC}^\dagger. \qquad (73)$$

Similarly to the case with the closed system transformation, (72) may preserve the set of the degrees of freedom associated with the RSF formalism in the system only if it does not depend on the conjugate field of the system, $(c_S, |\alpha_S^*\rangle)$. Close inspection of (72) reveals that this is possible only if (41) is fulfilled, which is what we wanted to prove.

Let us stress, however, that this condition is merely necessary for the RSF degrees of freedom to be preserved. Depending on the state of the bath, the remaining fields r_c, c_C, r_E, and c_E will, in general, cause the system to go beyond the RSF framework. In the most radical case, the equations may preserve the formalism's set of degrees of freedom only if all terms dependent on these additional fields vanish, reducing the system-environment ensemble to two separate closed systems.

Proof of Proposition 3

Finally, to prove Proposition 3, we note that, as is easy to calculate from their definitions, the initial reduced fields with the environment initially in the vacuum state fulfill

$$r_C = r_E = c_C = c_E = 0, \qquad |\alpha_E\rangle = |\alpha_E^*\rangle = 0. \qquad (74)$$

Plugging this into (72), we find that it simplifies to

$$r_S' = \mathcal{X}_{\uparrow S} r_S \mathcal{X}_{\uparrow S}^\dagger + \mathcal{X}_{\downarrow C} \mathcal{X}_{\downarrow C}^\dagger,$$

$$|\alpha_S'\rangle = \mathcal{X}_{\uparrow S} |\alpha_S\rangle, \qquad (75)$$

where we assumed the classicality condition (41). Clearly, the final field depends only on the initial RSF, preserving the associated degrees of freedom. Therefore, in this case, the condition (43) is not only necessary, but also sufficient for classicality.

It remains to show that if the transformation depends smoothly on time, the corresponding reduced kinetic equations are given by (43). In the time-dependent case, (75) becomes

$$r_S(t) = \mathcal{X}_{\uparrow S}(t)\, r_S(0)\, \mathcal{X}_{\uparrow S}^\dagger(t) + \mathcal{X}_{\downarrow C}(t) \mathcal{X}_{\downarrow C}^\dagger(t),$$

$$|\alpha_S(t)\rangle = \mathcal{X}_{\uparrow S}(t) |\alpha_S(0)\rangle. \qquad (76)$$

These relations are reversible

$$r_S(0) = \mathcal{X}_{\uparrow S}^{-1}(t) \left[r_S(t) - \mathcal{X}_{\downarrow C}(t) \mathcal{X}_{\downarrow C}^\dagger(t) \right] \mathcal{X}_{\uparrow S}^{-\dagger}(t),$$

$$|\alpha_S(0)\rangle = \mathcal{X}_{\uparrow S}^{-1}(t) |\alpha_S(t)\rangle. \qquad (77)$$

Differentiating (76) with respect to time, making use of (77), and rearranging the terms, we arrive at the differential evolution equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2}\Big[\mathcal{Y}_i, r\Big] + \frac{1}{2}\Big\{\mathcal{Y}_r, r\Big\} + \mathcal{W},$$

$$\frac{\mathrm{d}\,|\alpha\rangle}{\mathrm{d}t} = \frac{1}{2}\mathcal{Y}_i\,|\alpha\rangle + \frac{1}{2}\mathcal{Y}_r\,|\alpha\rangle\,,$$

(78)

where the matrices $\mathcal{Y}_r$, $\mathcal{Y}_i$, $\mathcal{W}$ are as defined in (44). Clearly, the derived equations have the form of the reduced kinetic equations characterized by (43). Thus, they describe valid dynamics provided the $\gamma_{\updownarrow}$ matrices are non-negative, as required by (42). This concludes the proof.

Appendix B: Proof of (60)

In this appendix, we derive the explicit forms of the operators (60) governing the reduced kinetic equations for the dynamical Casimir effect.

By comparing (54) with (36) and (40), we immediately identify

$$\mathcal{X}_{\uparrow S} = \mathrm{e}^{-\mathrm{i}\phi}f_{R+}, \qquad \mathcal{X}_{\downarrow C} = \mathrm{e}^{-\mathrm{i}\phi}f_{R-}. \tag{79}$$

Plugging this into (44) and then (43), on the way utilizing the differential equations (8), we obtain, after a lengthy but straightforward calculation,

$$h = \hbar\omega\left(\eta_+ + \eta_-\mathrm{Re}\Big[\frac{f_{R-}}{f_{R+}}\Big]\right) + \hbar\frac{\mathrm{d}\phi}{\mathrm{d}t},$$

$$\gamma_\uparrow = 2\omega\eta_-|f_{R-}|^2\left(\mathrm{Im}\Big[\frac{f_{R+}}{f_{R-}}\Big] + \mathrm{Im}\Big[\frac{f_{R-}}{f_{R+}}\Big]\right),$$

$$\gamma_\downarrow = 2\omega\eta_-\left[|f_{R-}|^2\,\mathrm{Im}\Big[\frac{f_{R+}}{f_{R-}}\Big] + (|f_{R-}|^2+1)\mathrm{Im}\Big[\frac{f_{R-}}{f_{R+}}\Big]\right].$$

(80)

It remains to show that these formulas reduce to (60).

In the case of the Hamiltonian, all we need to do is to differentiate (10) with respect to time. Due to the Leibniz integral rule,

$$\frac{\mathrm{d}\phi(\boldsymbol{k},t)}{\mathrm{d}t} = \omega(\boldsymbol{k})\delta(t)\beta(t)\cos\theta(\boldsymbol{k}), \tag{81}$$

from which we immediately see that the first lines of (80) and (60) coincide.

As for $\gamma_\uparrow$, we observe that for any complex number w

$$\mathrm{Im}\big[w^{-1}\big] = -\frac{\mathrm{Im}[w]}{|w|^2}. \tag{82}$$

Taking $w = f_{R+}/f_{R-}$, we get

$$\gamma_\uparrow = 2\omega\eta_-|f_{R-}|^2\left(1 - \frac{|f_{R-}|^2}{|f_{R+}|^2}\right)\mathrm{Im}\Big[\frac{f_{R+}}{f_{R-}}\Big]. \tag{83}$$

Using (16) and simplifying, we quickly find that the second lines of (80) and (60) also coincide.

Finally, we have to show that $\gamma_\downarrow = 0$. Once again utilizing the relation (16), we obtain

$$\gamma_\downarrow = 2\omega\eta_-|f_{R-}|^2\left(\mathrm{Im}\Big[\frac{f_{R+}}{f_{R-}}\Big] + \frac{|f_{R+}|^2}{|f_{R-}|^2}\mathrm{Im}\Big[\frac{f_{R-}}{f_{R+}}\Big]\right).$$

(84)

It is easy to see that the bracketed term vanishes upon the use of (82).

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