Abstract

In this doctoral thesis we are concerned with the construction and analysis of computational methods for linear equations, which oscillate with various frequencies. We present our ideas on the example of the linear Klein-Gordon equation with time- and space-dependent mass, of the form

$$\partial_t^2 \psi(\mathbf{x},t) \quad = \quad \Delta\psi(\mathbf{x},t) + f(\mathbf{x},t)\psi(\mathbf{x},t). \tag{3}$$

The form of the mass function $f(\mathbf{x},t)$ is a particularly interesting issue of this equations, as it can be characterized by oscillations of various frequencies, from low to extremely high. We assume that it can be expressed or approximated in the following form

$$f(\mathbf{x},t) \quad = \quad \alpha(\mathbf{x},t) + \sum_{|n|\leq N} a_n(\mathbf{x},t)\mathrm{e}^{\mathrm{i}\omega_n t}, \tag{4}$$

where $n \in \mathbb{N}$, $\omega_n \in \mathbb{R}$, $\omega_{\min} = \min_{n\leq N} |\omega_n| \geq 1$ and $\omega_{\max} = \max_{n\leq N} |\omega_n| < \infty$. Functions $\alpha(\mathbf{x},t)$ and $a_n(\mathbf{x},t)$, $n \leq N$, do not oscillate in $\omega_n s$ (i.e. their time derivatives are bounded independently of ω_n).

The function $f(\mathbf{x},t)$ can take the form of a non-oscillatory function (e.g. $\forall_n\, a_n(\mathbf{x},t) \equiv 0$), a highly-oscillatory function (e.g. $f(\mathbf{x},t) = a_2(\mathbf{x},t)\mathrm{e}^{i10^6 t}$) or a mixed function (e.g. $f(\mathbf{x},t) = \alpha(\mathbf{x},t) + a_1(\mathbf{x},t)\mathrm{e}^{it} + a_2(\mathbf{x},t)\mathrm{e}^{i10^6 t}$). Classical numerical methods, based on Taylor's expansion and quadratures, are very effective for non-oscillatory problems. However, in the presence of oscillations, they require restrictions on the time step size, namely $h < 1/\omega_{\max}$, which has a negative effect on their efficiency and computational cost, especially when the oscillations are very large.

The linear Klein-Gordon equation (3) is a representative model, on which we will propose and discuss various approaches to the computational analysis of the problem of oscillatory input term $f(\mathbf{x},t)$, including asymptotic expansion, variation of constants, Magnus expansion and decomposition of the vector field.

The motivation for our research on the equation (3) was the paper [Bader et al., 2019], in which the computational methods for such an equation were presented for the first time. The numerical schemes proposed there work only in case of the non-oscillatory function $f(\mathbf{x},t)$. The problems of the classical methods and those presented in the paper [Bader et al., 2019], resulting from the presence of high oscillations in the function f, seemed very interesting to us, therefore we proposed three other numerical approaches in the presence of different oscillations in the Klein-Gordon equation (3).

In the first presented method, we assume that the solution of the Klein-Gordon equation (3) may take the form of an infinite modified Fourier series. As a result we get numerical-asymptotic method, which accuracy increases with the increasing magnitude of oscillation frequencies ω_n. The method was designed for the case, when we have only high frequencies in function $f(\mathbf{x},t)$

The second method presented in the thesis is a third-order in time method which is equally efficient in the full range of frequencies, e.g. $f(x,t) = a_0(x,t)$, $mf(x,t) = a_2(x,t)\mathrm{e}^{i10^6 t}$ and $f(x,t) = a_0(x,t) + a_1(x,t)\mathrm{e}^{it} + a_2(x,t)\mathrm{e}^{i10^6 t}$, where the error constant does not grow with the size of ω_ns, and no ratio between time step h and $\omega_{\max}$ needs to be imposed. This goal was achieved by rewriting the Klein-Gordon equation (3) in the form of a non-homogeneous system of ordinary differential equations and by application of the well-known variation of constants method, also known as Duhamel's formula. Moreover, the use of the Filon method for approximation of highly oscillatory integrals, which are part of the numerical scheme, makes the time step size h independent of the magnitude of oscillations ω_n. We will present a rigorous proof of convergence, discuss the structure of the error of the method and explain why it is not affected by possibly extremely high oscillations.

In the third presented numerical method, we use the Magnus expansion and various decompositions of the vector field, which are well-known methods of approximation of the solution of evolution equations. Unlike the method based on Duhamel's formula, we will not prove the global order of convergence, but will focus on local accuracy by determining the leading error terms appearing during the derivation of the

method. We will show that the local accuracy of the method is described by the expressions

$$\mathcal{O}(h^5 + \min\{h^3, \frac{h^2}{\omega_{\min}}, h^5\omega_{\max}^2\}) \quad \text{for} \quad \alpha(\mathbf{x}, t) \neq 0$$

and

$$\mathcal{O}(\min\{h^3, \frac{h^2}{\omega_{\min}}, h^5\omega_{\max}^2\}) \quad \text{for} \quad \alpha(\mathbf{x}, t) \equiv 0,$$

which depend on the relationship between time step h and $\omega_{\min}$, $\omega_{\max}$. This method works effectively for all magnitudes of oscillation in function $f(\mathbf{x}, t)$ and shows up the highest accuracy among the methods proposed so far for the linear Klein-Gordon equation with time- and space-dependent mass.

The methods are described in separate chapters, and each chapter ends with a presentation of the results of numerical simulations and a comparison to the methods presented before.